# Calculation of electric field around a conducting cone illuminated by a plane wave

## I. USED FORMULAS

Formulas were taken from a book [1]. Firstly, I calculated arrays of positive numbers $p$ and $q$ by solving equations (18.192),

$$\begin{aligned} P_p^{-m}(\cos\theta_1) &= 0, \\ \frac{\partial}{\partial\theta} P_p^{-m}(\cos\theta_1) &= 0. \end{aligned} \tag{1}$$

Here, $\theta_1$ equals 180° minus half-angle of the cone. For illustration, obtained data for the cone angle 90° are shown on Fig. 1.

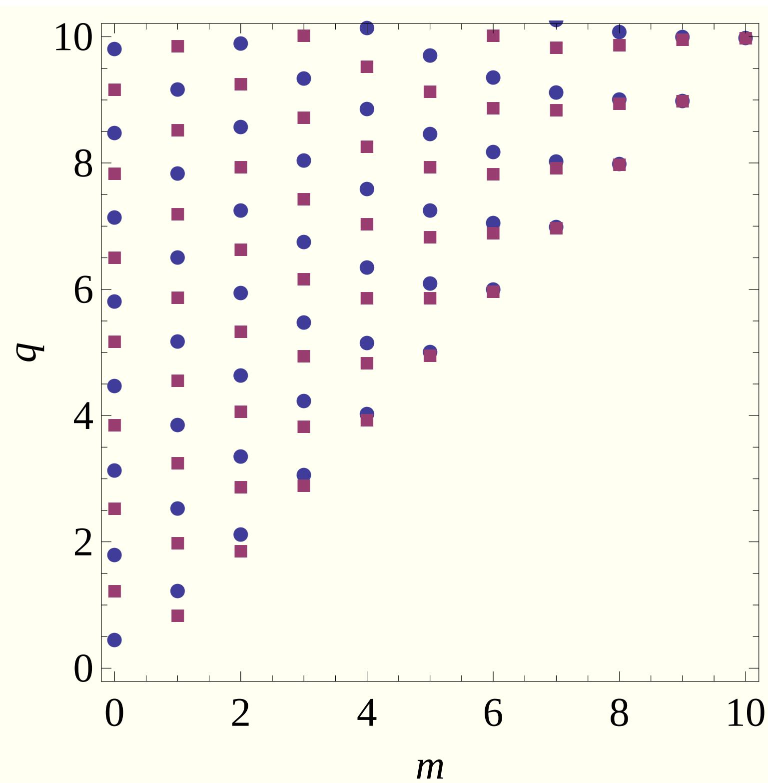


FIG. 1. Roots of Legendre functions and their derivative, equation (1), for increasing values of $m$. Circles are values of $p$, and squares are values of $q$. Cone angle equals 90°

In order to calculate electric field, I calculated two functions, $u(\theta, \phi, r)$ and $v(\theta, \phi, r)$ by formulas (18.188) and (18.189) from the book [1]. There, I limited summation over $m$ up to some $m_{\max}$, and summation over roots $p$ and $q$ up to some maximum root number $n_{\max}$, so final formulas are

$$\begin{aligned} u &= \frac{2i}{k\sin\theta_1}\sum_{m=0}^{m_{\max}}\varepsilon_m\sum_{n=1}^{n_{\max}}\frac{2p_n+1}{p_n(p_n+1)}\frac{e^{\frac{1}{2}ip_n\pi}j_{p_n}(kr)P^m_{p_n}(\cos\theta)}{\frac{\partial}{\partial\theta}P^m_{p_n}(\cos\theta_1)\frac{\partial}{\partial p}P^m_{p_n}(\cos\theta)} \\ &\times\left[m\sin m(\phi-\phi_0)\cos\beta\frac{P^m_{p_n}(\cos\theta_0)}{\sin\theta_0}+\cos m(\phi-\phi_0)\sin\beta\frac{\partial}{\partial\theta}P^m_{p_n}(\cos\theta_0)\right], \\ v &= \frac{-2i}{k\sin\theta_1}\sum_{m=0}^{m_{\max}}\varepsilon_m\sum_{n=1}^{n_{\max}}\frac{2q_n+1}{q_n(q_n+1)}\frac{e^{\frac{1}{2}iq_n\pi}j_{q_n}(kr)P^m_{q_n}(\cos\theta)}{P^m_{q_n}(\cos\theta_1)\frac{\partial^2}{\partial q\partial\theta}P^m_{q_n}(\cos\theta_1)} \\ &\times\left[\cos m(\phi-\phi_0)\cos\beta\frac{\partial}{\partial\theta}P^m_{q_n}(\cos\theta_0)\sin\theta_0-m\sin m(\phi-\phi_0)\sin\beta\frac{P^m_{q_n}(\cos\theta_0)}{\sin\theta_0}\right]. \end{aligned} \tag{2}$$

In equation (2), $\varepsilon_m$ is 1 for $m=0$ and 2 otherwise, $\beta$ determines polarization of light ($\beta=0$ parallel to the cone axis and $\beta=\pi/2$ perpendicular), angles $(\theta_0,\phi_0)$ determine direction of incident light, and $(\theta,\phi,r)$ is the location of observation point.

Three components of the electric field at a point $(\theta,\phi,r)$ are expressed through functions $u(\theta,\phi,r)$ and $v(\theta,\phi,r)$ and their derivatives according to formula (18.183) from the book [1],

$$\begin{aligned} E_\theta &= \frac{ik}{\sin\theta}\frac{\partial v}{\partial\phi}+\frac{1}{r}\frac{\partial(ru)}{\partial r}, \\ E_\phi &= -ik\frac{\partial v}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial^2(ru)}{\partial r\partial\phi}, \\ E_r &= \frac{\partial^2(ru)}{\partial r^2}+k^2ru. \end{aligned} \tag{3}$$

Here as usually, absolute value of a complex number means the amplitude of electric field, and argument means the phase.

Near the tip of the cone ($kr\ll 1$), electric field could be determined by a one-term formula (18.195) from the book [1],

$$\begin{aligned} E_r &= \frac{ie^{-\frac{1}{2}ip_1\pi}}{\sin\theta_1}\frac{\sqrt{\pi}(kr)^{p_1-1}}{2^{p_1-1}\Gamma\left(p_1+\frac{1}{2}\right)}\frac{P_{p_1}(\cos\theta)P^1_{p_1}(\cos\theta_0)\sin\beta}{P^1_{p_1}(\cos\theta_1)\frac{\partial}{\partial p}P_{p_1}(\cos\theta_1)}, \\ E_\theta &= \frac{ie^{-\frac{1}{2}ip_1\pi}}{\sin\theta_1}\frac{\sqrt{\pi}(kr)^{p_1-1}}{2^{p_1-1}\Gamma\left(p_1+\frac{1}{2}\right)}\frac{\frac{\partial}{\partial\theta}P_{p_1}(\cos\theta)P^1_{p_1}(\cos\theta_0)\sin\beta}{P^1_{p_1}(\cos\theta_1)\frac{\partial}{\partial p}P_{p_1}(\cos\theta_1)}, \\ E_\phi &= 0, \end{aligned} \tag{4}$$

where $p_1$ is a zero of the function $P_p(\cos\theta_1)$ on the interval $(0,1)$.

## II. RESULTS OF CALCULATIONS

Here, we always consider the case when the direction of incident light is perpendicular to the cone axis ($\theta_0 = \pi/2$). On plots, we show mean-squared electric field $\bar{E} \equiv \sqrt{|E_\theta|^2 + |E_\phi|^2 + |E_r|^2}$. For simplicity, we plotted results only along the line going through the tip of the cone and parallel to direction of incident light, i.e. $\theta = \pi/2$ and $\phi = 0$ in front of the cone or $\phi = \pi$ behind it.

We found that truncated-series results given by equation (3) are sufficiently convergent if the upper limits of summation, $m_{\max}$ and $n_{\max}$ exceed value of $r/\lambda$ times 10. In our calculations, we used $m_{\max} = m_{\max} = 20$.

### A. Cone angle 90°

On Fig. 2, we plotted the mean electric field both in front of the cone ($r > 0$) and behind the cone ($r < 0$) for polarization perpendicular to the cone axis. In this case, $E_\theta = E_r = 0$, and $E_\phi$ oscillates in front of the cone because of interference of two possible paths, one direct, and second path reflected from the cone. At $r \to 0$, there is a node of electric field, $E \to 0$. According to one-term approximate formula (4), $E = 0$ identically.

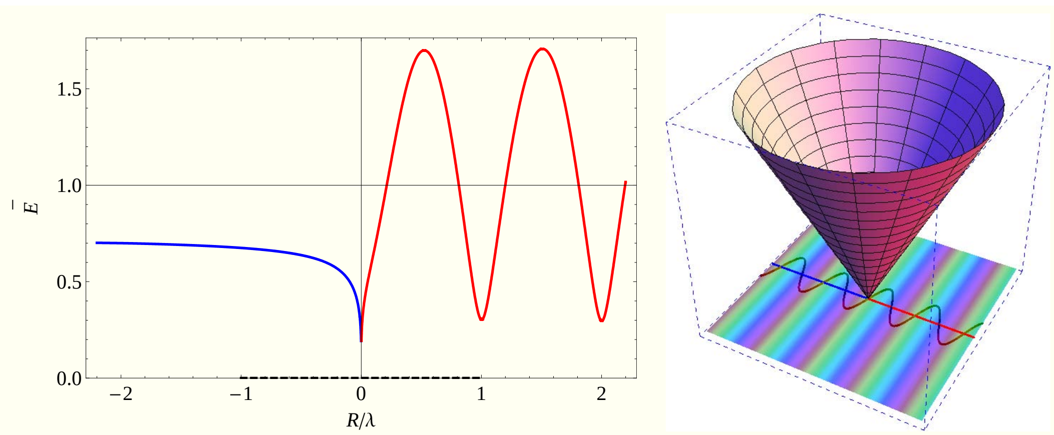


FIG. 2. Mean electric field along the line going through the tip of a cone and parallel to direction of the incident light. Red half-line is the area in front of the cone, and blue line behind it. Polarization of light is perpendicular to the cone axis. Dashed line which is identically zero is the result of using equation (4) which is valid only in the vicinity of the tip. Cone angle equals 90°

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On Fig. 3, the results are plotted for polarization parallel to the cone axis. In this case,

$E_\phi = 0$, but both $E_\theta$ and $E_r$ are non-zero. Oscillation are less regular because of additional interplay of two non-zero components. According to one-term approximate formula (4), $E = 0.46|r|^{-0.54}$, which appears to be accurate at $|r| < 0.2\lambda$. Here, electric field tends to infinity on the tip because of infinite curvature of the cone's tip and induced electric charges at the tip. Notice, that only the component of electric field which is parallel to the cone's axis is responsible for induction of the charge, so there is no such effect for perpendicular polarization.

At finite, but small $r$ the electric field may be very large. For example, for $r = \pm 0.6 \cdot 10^{-6}\lambda$, the electric field equals to 1000.

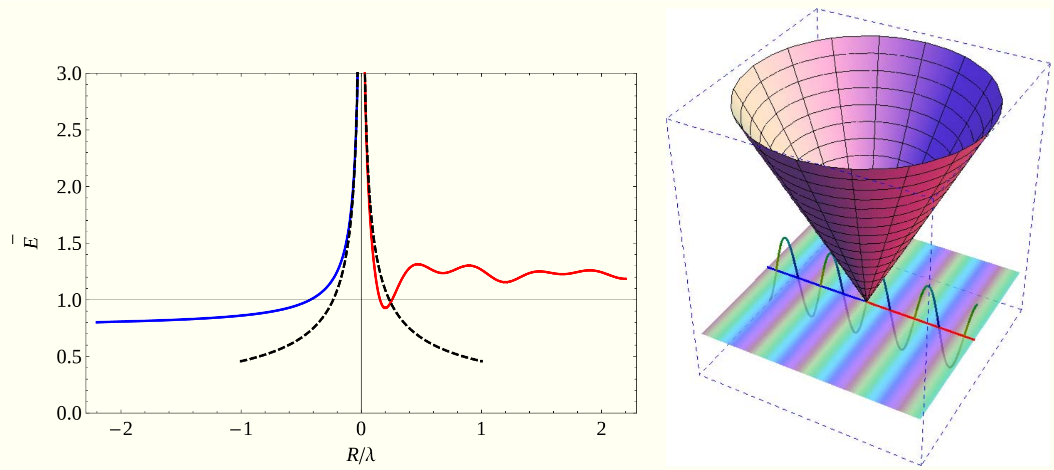


FIG. 3. Mean electric field along the line going through the tip of a cone and parallel to direction of the incident light. Red half-line is the area in front of the cone, and blue line behind it. Polarization of light is parallel to the cone axis. Dashed line is the result of using equation (4) which is valid only in the vicinity of the tip. Cone angle equals 90°

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### B. Cone angle 40°

Here, we consider a cone with sharper angle at the tip.

On Fig. 4, we plotted the mean electric field for polarization perpendicular to the cone axis. In this case, $E_\phi$ oscillates more rapidly because of larger difference between two possible paths of light. In the very nearest vicinity of the tip, electric field goes to zero.

On Fig. 5, the results are plotted for polarization parallel to the cone axis. According to one-term approximate formula (4), $E = 0.16|r|^{-0.73}$, which appears to be accurate at

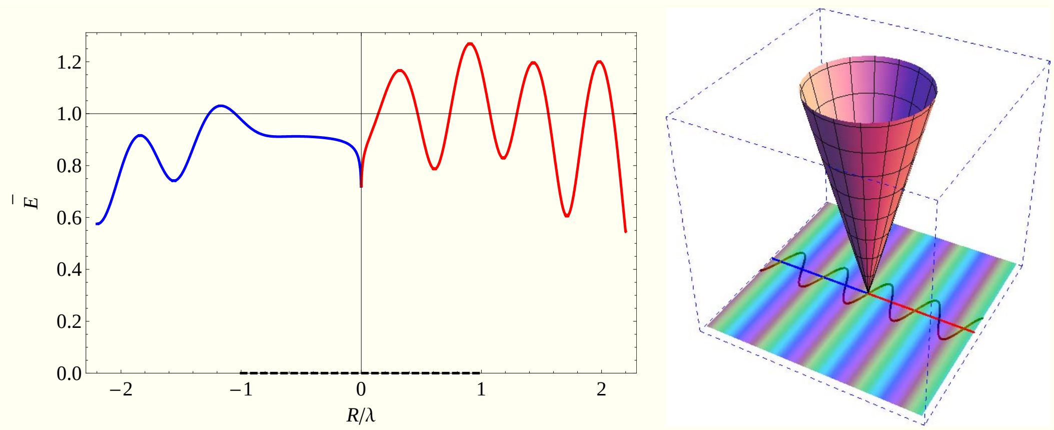


FIG. 4. Mean electric field along the line going through the tip of a cone and parallel to direction of the incident light. Red half-line is the area in front of the cone, and blue line behind it. Polarization of light is perpendicular to the cone axis. Dashed line which is identically zero is the result of using equation (4) which is valid only in the vicinity of the tip. Cone angle equals 40°

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$|r| < 0.1\lambda$. For $r = \pm 6 \cdot 10^{-6}\lambda$, the electric field equals to 1000.

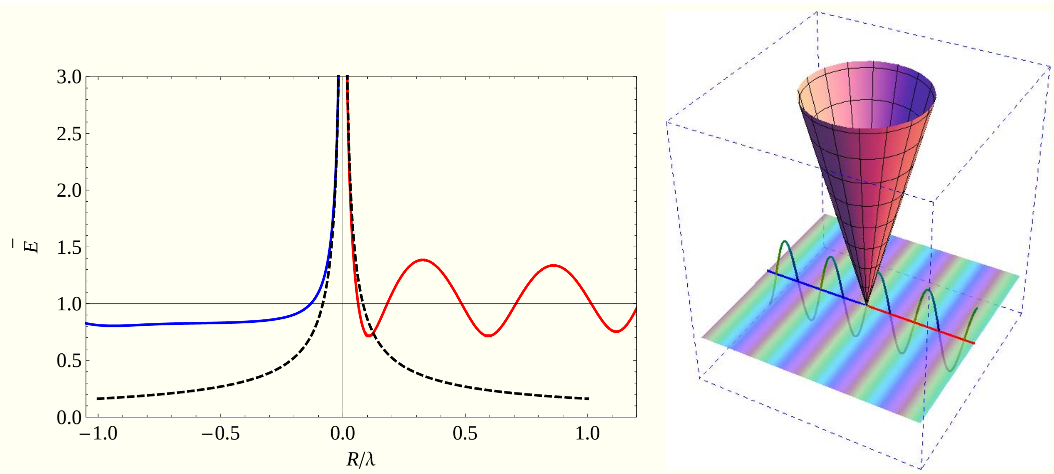


FIG. 5. Mean electric field along the line going through the tip of a cone and parallel to direction of the incident light. Red half-line is the area in front of the cone, and blue line behind it. Polarization of light is parallel to the cone axis. The field at the tip goes to infinity as a result of induced charges. Dashed line is the result of using equation (4) which is valid only in the vicinity of the tip. Cone angle equals 40°

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### C. Discussion

Theoretically, the field on the tip should approach infinity $\sim |r|^{p_1-1}$ where $0 < p_1 < 1$ and $P_{p_1}(\cos\theta_1) = 0$. In practice, there is finite penetration length because of finite conductance of the metal. As a result, the cone's tip is transparent for light when the distance to the tip is smaller than $\lambda/n_i$, where $n_i$ is imaginary part of refraction index. Since it is typically larger than $\lambda/10$, this effect cannot be observed in practice.

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[1] J. J. Bowman, in J. J. Bowman, T. B. A. Senior, P. L. E. Uslenghi, *Electromagnetic and Acoustic Scattering*, Summa Book, 1987, Chapter 18.