Bohr, scaling and more
(stable circular orbits for helium atom)

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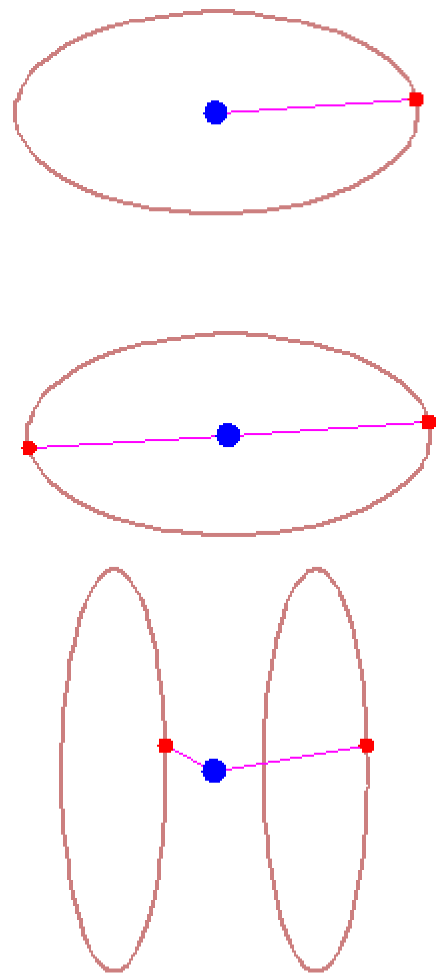


Hydrogen atom

Stable orbit

Helium atom

Unstable orbits

? Stable orbits ?

J. A. West et al., Classical limit states of the helium atom. Phys. Rev. A, 58, 186, 1998

The logical progression of the hydrogenic studies is to extend them to include planetary atoms with multiple valence electrons [2–6]. However, even for the simplest such atom, helium, this extension is nontrivial because the old quantum theory of Bohr was never successfully modified to include helium. Early in this century a considerable effort was made to develop a classical model for helium, but no stable planetary orbits were found [see Figs. 1(a) and 1(b)]. By 1920 Bohr had concluded that for stability, one must allow for ‘‘possibilities of more complicated motions,’’ [7] but before these possibilities could be explored, classical atomic physics was abandoned in the wake of wave mechanics and classical helium was put aside.

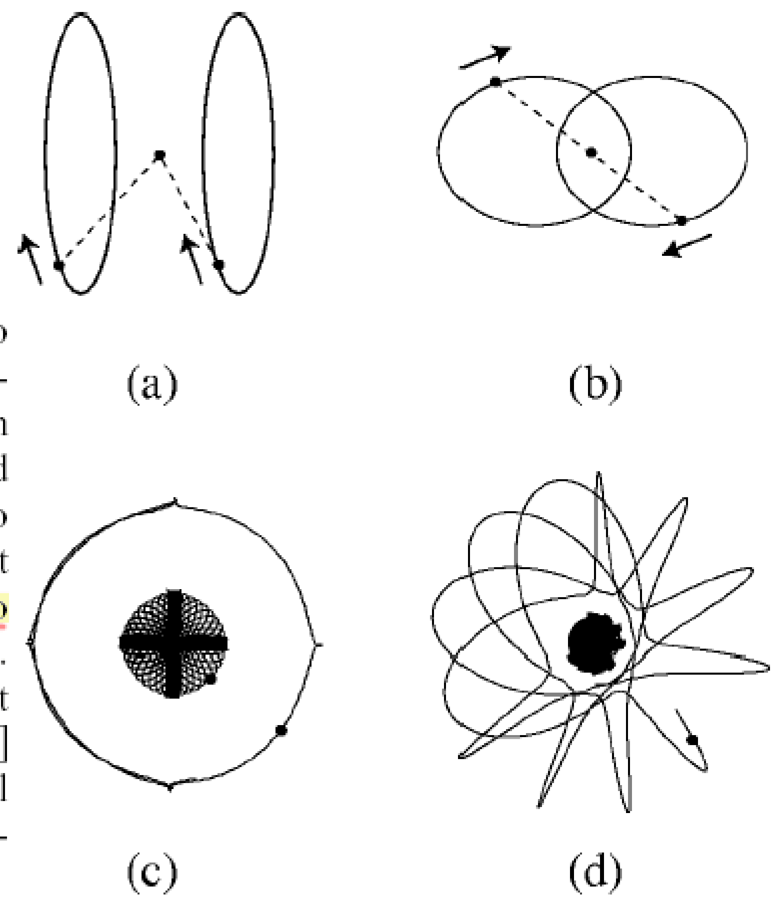


FIG. 1. A pictorial survey of some classical helium orbits. The two-electron trajectories shown in (a) and (b) are highly symmetric unstable orbits which were studied in an attempt to extend the Bohr model [3,7,8]. The orbits shown in (c) and (d) are stable orbits in which the dynamics of the individual electrons is quite dissimilar. In these orbits it is difficult to resolve the rapid motion of the inner electron.

Interdimensional degeneracies

Arbitrary D

$$\left(-\frac{1}{2}\vec{\nabla}_D^2-\frac{1}{r}-E_D\right)\Psi_{D,0}(\vec{r})=0 \qquad \Psi_{D,0}(\vec{r})=r^{-(D-1)/2}\psi_D(r)$$

$$\left(-\frac{1}{2}\frac{d^2}{dr^2}+\frac{(D-1)(D-3)}{8r^2}-\frac{1}{r}-E_D\right)\psi_D(r)=0$$

$D=2$

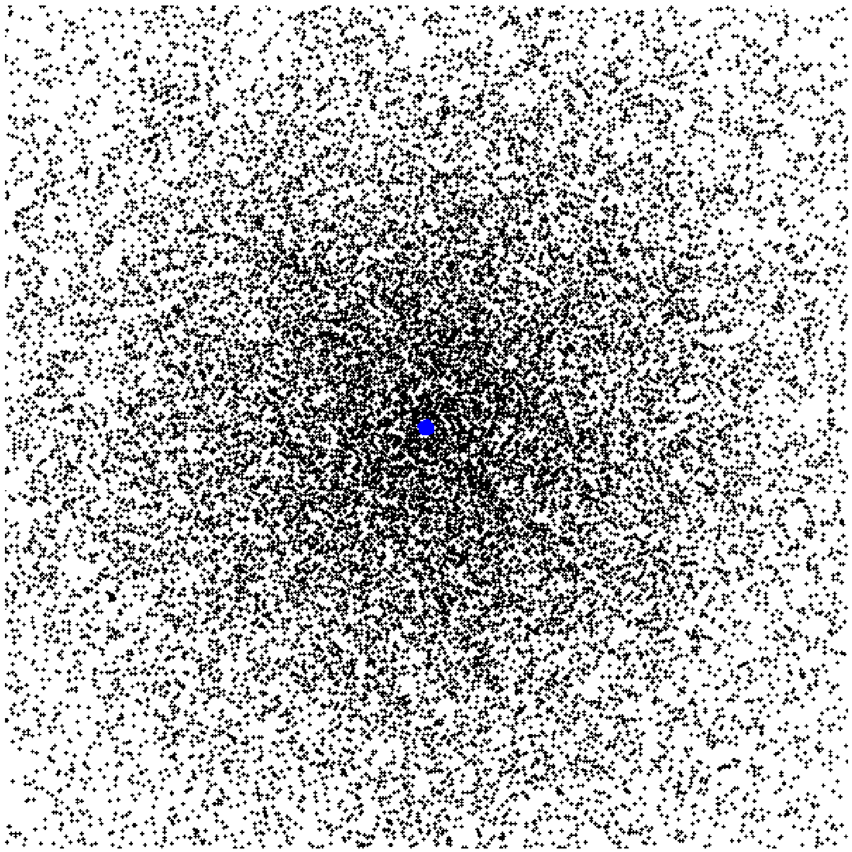
$$\left(-\frac{1}{2}\vec{\nabla}^2-\frac{1}{r}-\mathcal{E}_m\right)\Psi_{2,m}(\vec{r})=0 \qquad \Psi_{2,m}(\vec{r})=r^{-1/2}e^{im\varphi}\psi_m(r)$$

$$\left(-\frac{1}{2}\frac{d^2}{dr^2}+\frac{m^2-1/4}{2r^2}-\frac{1}{r}-\mathcal{E}_m\right)\psi_m(r)=0$$

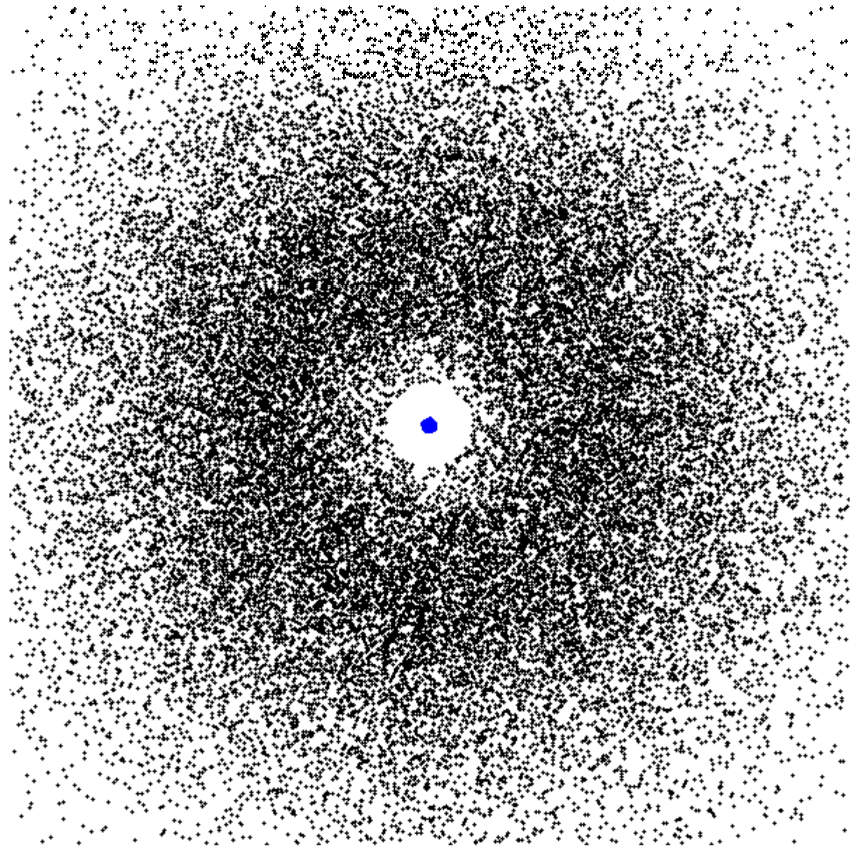
$$\left(m-\frac{1}{2}\right)\left(m+\frac{1}{2}\right)=\left(\frac{D-3}{2}\right)\left(\frac{D-1}{2}\right) \qquad \begin{pmatrix} D \\ L=0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} D=2 \\ m=(D-2)/2 \end{pmatrix}$$

Circular Rydberg states as the limit of $D \to \infty$

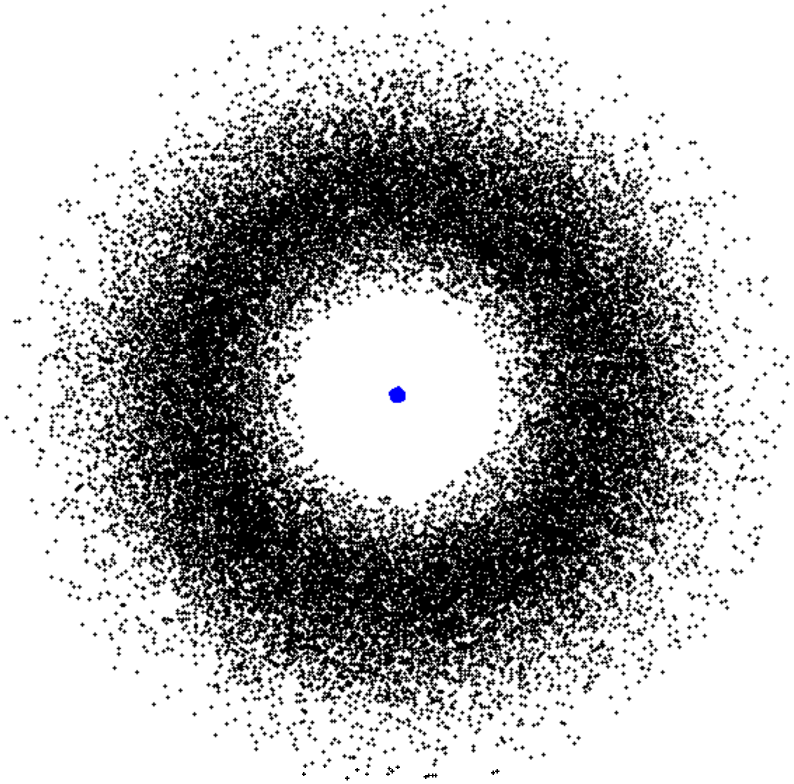
$$\psi_m(r) = N_m \, r^{m+1/2} \exp\left(-\frac{r}{m+1/2}\right), \quad m = D/2 - 1$$



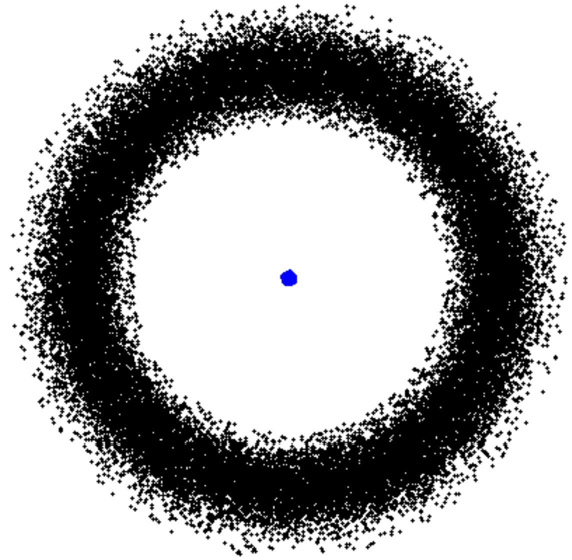
$D = 2$



$D = 5$

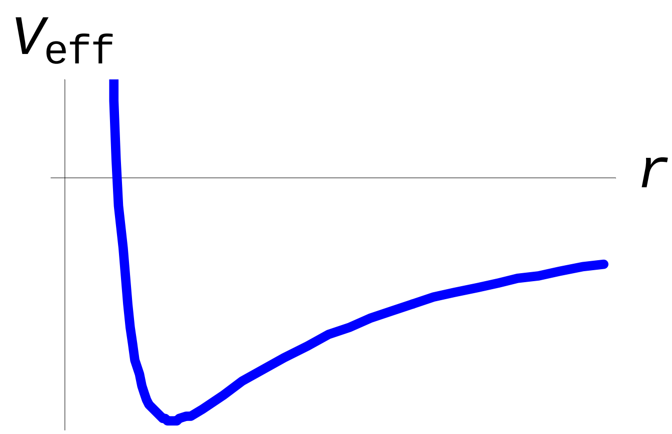


$D = 20$



$D = 100$

$$V_{\text{eff}} = \frac{m^2 - 1/4}{2r^2} - \frac{1}{r}$$



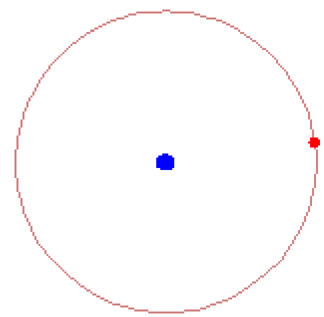
Classical mechanics as the limit of $D \to \infty$

$$\tilde{\psi}(\tilde{r}) = \psi\left((m^2 - 1/4)^{1/2}\tilde{r}\right) \qquad \tilde{E} = \left(m^2 - 1/4\right)\mathcal{E}$$

$$\left[-\frac{1}{2(m^2 - 1/4)}\frac{d^2}{d\tilde{r}^2} + \frac{1}{2\tilde{r}^2} - \frac{1}{\tilde{r}} - \tilde{E}\right]\tilde{\psi}(\tilde{r}) = 0$$

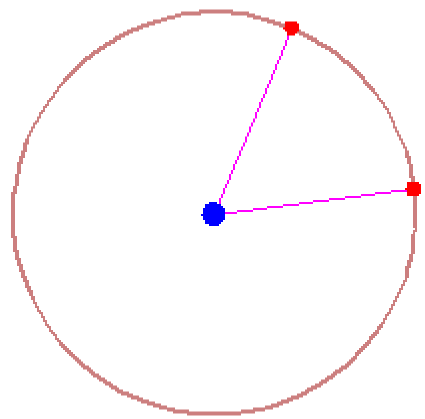
$$\hbar \leftrightarrow \left(m^2 - 1/4\right)^{-1/2}$$

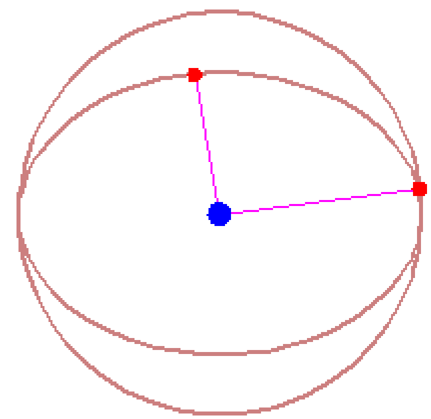
$$\hbar \to 0$$



Unstable orbits for helium in two and three dimensions

The case of non-interacting electrons



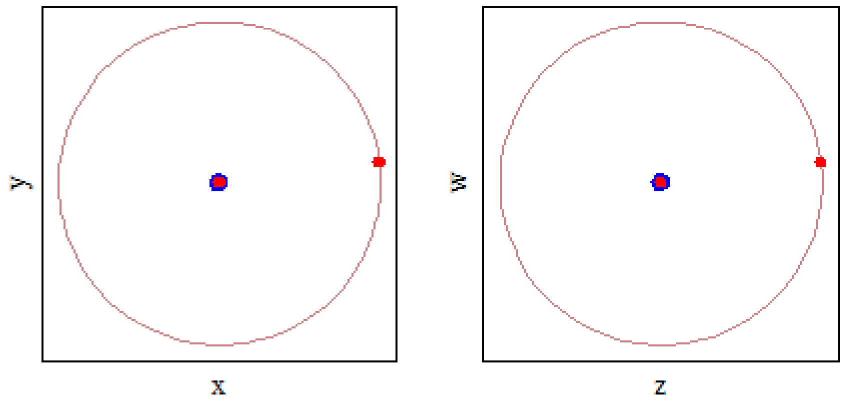


Stable orbits for helium in four dimensions (x, y, z, w)

The case of non-interacting electrons

$$1/r_{12} \to g/r_{12}, \quad g = 0$$

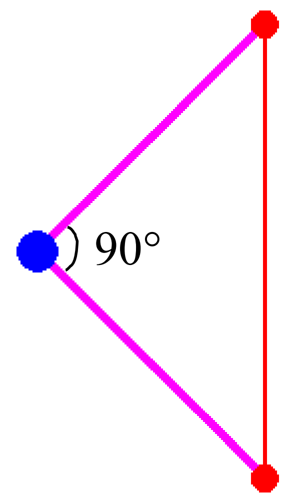
$$
\begin{aligned}
x_1(t) &= R\cos\omega t & x_2(t) &= 0 \\
y_1(t) &= R\sin\omega t & y_2(t) &= 0 \\
z_1(t) &= 0 & z_2(t) &= R\cos\omega t \\
w_1(t) &= 0 & w_2(t) &= R\sin\omega t
\end{aligned}
$$



$$r_1(t) = \left(x_1^2 + y_1^2 + z_1^2 + w_1^2\right)^{1/2} = R$$

$$r_2(t) = \left(x_2^2 + y_2^2 + z_2^2 + w_2^2\right)^{1/2} = R$$

$$r_{12}(t) = \left((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (w_1 - w_2)^2\right)^{1/2} = \sqrt{2}R$$

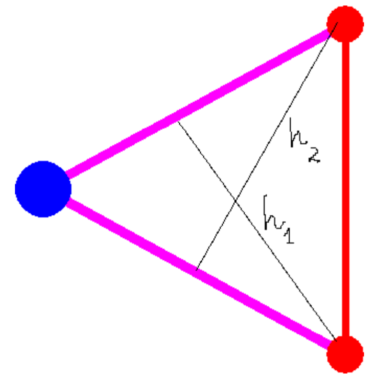


Turning on repulsion between electrons

$$L_{xy} = x_1 p_{y_1} - y_1 p_{x_1} + x_2 p_{y_2} - y_2 p_{x_2} = L$$

$$L_{zw} = z_1 p_{w_1} - w_1 p_{z_1} + z_2 p_{w_2} - w_2 p_{z_2} = L$$

$$L_{xz} = L_{xw} = L_{yz} = L_{yw} = 0$$

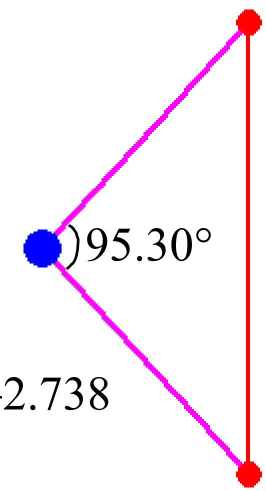


$$T = \frac{1}{2}\left(p_{x_1}^2 + p_{y_1}^2 + p_{z_1}^2 + p_{w_1}^2 + p_{x_2}^2 + p_{y_2}^2 + p_{z_2}^2 + p_{w_2}^2\right)$$

$$= \frac{L^2}{2}\left(\frac{1}{h_1^2} + \frac{1}{h_2^2}\right) + \frac{1}{2}\left(p_{x_1}^2 + p_{z_1}^2 + p_{x_2}^2 + p_{z_2}^2\right)$$

$$E = T + V = V_{\text{eff}}(r_1, r_2, r_{12})$$

$$V_{\text{eff}}(r_1, r_2, r_{12}) = -\frac{2}{r_1} - \frac{2}{r_2} + \frac{1}{r_{12}} + \frac{L^2}{2}\left(\frac{1}{h_1^2} + \frac{1}{h_2^2}\right)$$



$E_{\min} = -2.738$

One electron in central field

$$L_{xy} = xp_y - yp_x = L$$

$$x = r, \quad y = 0$$

$$p_y = L / r$$

$$T = \frac{1}{2}\left(p_x^2 + p_y^2\right) = \frac{L^2}{2r^2} + \frac{1}{2} p_x^2$$

Invariants of motion

$$L^2 = \frac{1}{4} L_{ij} L_{ij}$$

$$M = \frac{1}{4} L_{ij} \overline{L}_{ij}$$

$$\overline{L}_{ij} = \frac{1}{2} \varepsilon_{ijkl} L_{kl}$$

$$K_{\pm} = \frac{1}{8} \left(L_{ij} \pm \overline{L}_{ij}\right)\left(L_{ij} \pm \overline{L}_{ij}\right) = L^2 \pm M$$

$$-L^2 \leq M \leq L^2$$

$$M = \frac{1}{2} \varepsilon_{ijkl} r_{1i} p_{1j} r_{2k} p_{2l}$$

In our case

$$M = L^2$$

$$L_{ij} = \overline{L}_{ij}$$

Sergeev A. V. (1989): 1/N-expansion for the three-body problem. Sov. J. of Nucl. Phys. **50**, 589

Conclusions

- In the limit of large D, the energy of a hydrogen atom equals to the classical energy in two-dimensional Bohr model (circular orbit).

- The energy of a two-electron atom equals to the classical energy of four-dimension generalization of Bohr model

- In the generalized Bohr model, electrons move on stable circular orbits and form rigid triangle configuration