Positron–nucleus resonances in electric and magnetic fields

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Resonance states of a positron in the Coulomb field of a nucleus and in parallel homogeneous electric and magnetic fields are studied. In the classical description, the positron undergoes oscillatory motion along the symmetry axis, being reflecting consecutively from the nucleus and the anode. The magnetic field does not permit the positron to go around the nucleus and move to the cathode, stabilizing the resonances. The nucleus is regarded as infinitely heavy, and the problem reduces to a one-particle Schrödinger equation. The energies and widths of the resonances are determined by means of versions of perturbation theory in weak electric and strong magnetic fields, the $1/n$ expansion, and the quasiclassical approximation. Different methods are in good agreement with one another.

1. A positively charged nucleus acts on the positron as a repulsive Coulomb center; therefore, such a system does not have bound states. However, in the presence of external fields, including the field of another nucleus,[1)] there are quasistationary states. In the present paper we study the case of homogeneous parallel electric ($\vec{\mathscr{E}}$) and magnetic ($\vec{\mathscr{H}}$) fields. Even though there exist many papers on the motion of an electron in the nuclear field and in the fields $\vec{\mathscr{E}}$ and $\vec{\mathscr{H}}$, the motion of a positron has not yet been considered. Such a model, being one of the simplest ones, deserves theoretical attention.

The pe^+ system is not electrically neutral and rapidly disintegrates while being attracted to the cathode. Therefore, in positron scattering on protons in the $\vec{\mathscr{E}}\|\vec{\mathscr{H}}$ fields, pe^+ resonances will arise only for a very short time. Experimentally, the transitions between different quasistationary levels will produce individual lines in the radio-frequency spectrum.

In Sec. 2 we obtain the asymptotic behavior of the energy in a weak electric field, and in Sec. 3 we obtain that behavior in a strong magnetic field. In Sec. 4 we make use of the $1/n$ expansion (for the states that in the limit $n\to\infty$ are described by the classical motion of the positron along a circular orbit). In Sec. 5 we apply the quasiclassical quantization rules for a purely electric field ($\mathscr{H}=0$).

2. Assuming that the nucleus in infinitely heavy, let us solve the one-particle Schrödinger equation with the potential

$$V(\rho,\ z)=1/r+\mathscr{E}z+{}^1/_8\mathscr{H}^2\rho^2, \qquad (1)$$

where ρ and z are the cylindrical coordinates, $r=(\rho^2+z^2)$,$^{1/2}$ the fields $\vec{\mathscr{E}}$ and $\vec{\mathscr{H}}$ are directed along the z axis, and the atomic system of units $\hbar=m_e=e=1$ is used. The nuclear charge is set equal to unity, which can always be achieved by a scale transformation. In Eq. (1) we have dropped the paramagnetic term $\frac{1}{2}\mathscr{H}(L_z+2S_z)$, whose contribution to the energy is trivial.

In the classical approximation the positron can either be at rest at the point of an extremum of the potential (1) on the symmetry axis at $z=z_0=\mathscr{E}^{-1/2}$ or it can oscillate along the z axis between the nucleus and the anode. In a sufficiently strong magnetic field $\mathscr{H}>2\mathscr{E}^{3/4}$, the point ($\rho_0=0,z_0$) becomes a local minimum of the potential, and the stability condition is fulfilled. The electric and magnetic fields in the vicinity of the point (ρ_0,z_0) form a resonator.

A lower bound on the energy of the resonances is given by the minimum of the potential

$$E_0=V(\rho_0,\ z_0)=2\mathscr{E}^{1/2}.$$

An upper bound on the energy, above which in the classical limit the resonance can decay, is given by the value of the potential at the saddle point:

$$E_s=V(\rho_s,\ z_s)={}^3/_2(\mathscr{H}/2)^{2/3}+2\mathscr{E}^2/\mathscr{H}^2,$$

where

$$\rho_s=2\mathscr{H}^{-1}[(\mathscr{H}/2)^{2/3}-4\mathscr{E}^2/\mathscr{H}^2]^{1/2},\quad z_s=4\mathscr{E}/\mathscr{H}^2.$$

Let us represent the potential in the form $V=V_0+W$, where

$$V_0(\rho,\ s)=2\mathscr{E}^{1/2}+{}^1/_8\mathscr{H}^2\rho^2+[-{}^1/_2\rho^2+(z-z_0)^2]\mathscr{E}^{3/2}$$

is the harmonic-oscillator potential summed with the energy E_0, and

$$W(\rho,\ z)=[{}^3/_2\rho^2-(z-z_0)^2](z-z_0)\mathscr{E}^2+[{}^3/_8\rho^4-3(z-z_0)^2\rho^2+(z-z_0)^4]\mathscr{E}^{5/2}+O(\mathscr{E}^3) \qquad (2)$$

is the perturbing potential. In the harmonic-oscillator approximation we have

$$E=2\mathscr{E}^{1/2}+(2n_\rho+m+1)\omega_\rho+(n_z+{}^1/_2)\omega_z,$$

where $\omega_\rho=\alpha\mathscr{E}^{3/4}$ and $\omega_z=\sqrt{2}\mathscr{E}^{3/4}$ are the oscillator frequencies; n_z, n_ρ, and $m=0,\ 1,\ 2,\ldots$ are the quantum numbers; and

$$\alpha=(\mathscr{H}^2/4\mathscr{E}^{3/2}-1)^{1/2}. \qquad (3)$$

Taking into account the nonharmonic character of (2), we find

$$E=2\mathscr{E}^{1/2}+\left(\alpha N+\frac{\sqrt{2}}{2}p\right)\mathscr{E}^{3/4}+\left\{\frac{1}{64}(5-3p^2)-\frac{3}{32\alpha(2\alpha^2-1)}\times\left[4\sqrt{2}(1+\alpha^2)pN+\frac{1+4\alpha^2}{\alpha}(m^2-1)-\frac{3}{\alpha}N^2\right]\right\}\mathscr{E}+O(\mathscr{E}^{5/4}), \qquad (4)$$

where, for brevity, we have introduced the notation

$$N=2n_\rho+m+1,\quad p=2n_z+1. \qquad (5)$$

It is clear that Eq. (4) is an asymptotic expansion of the

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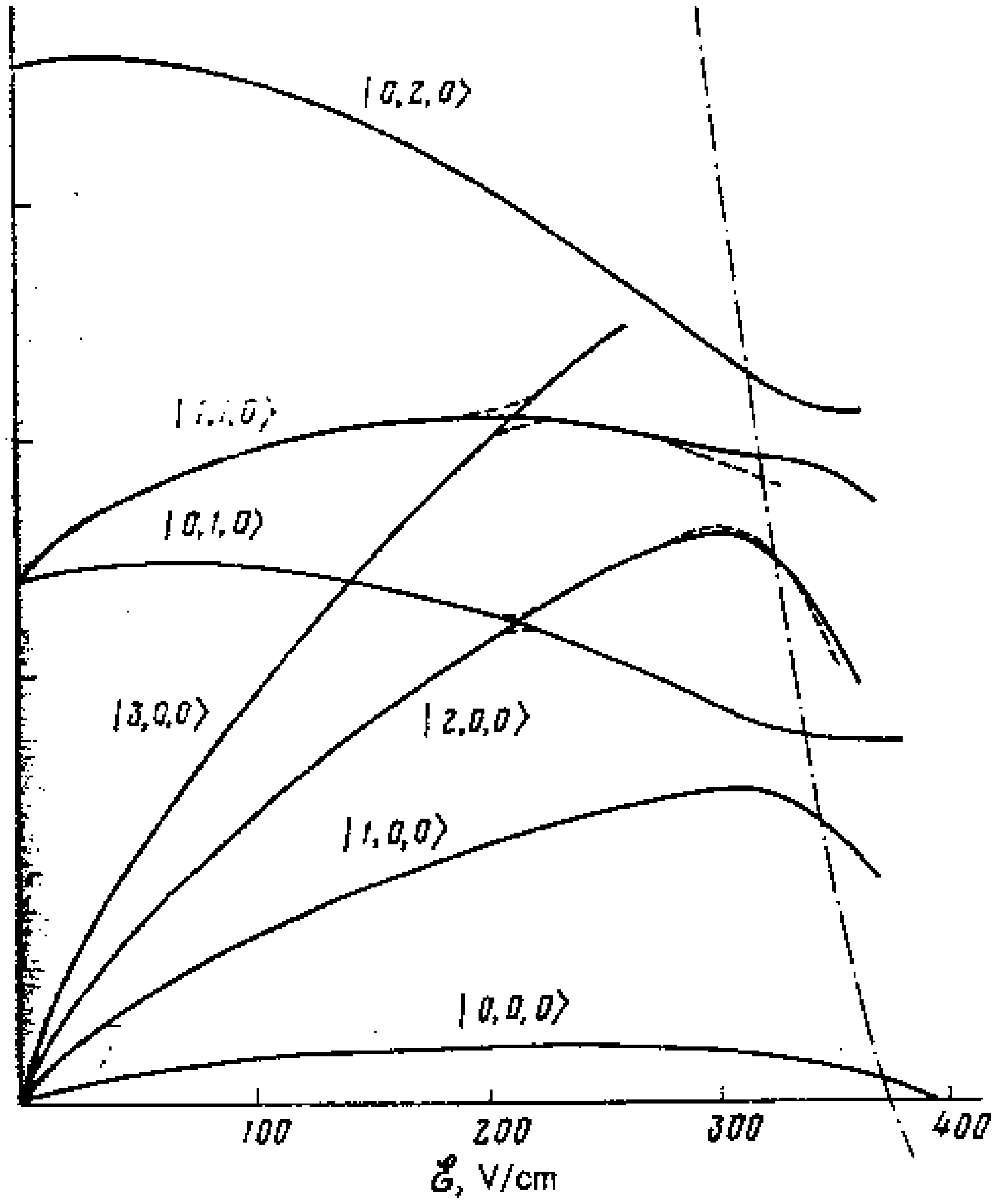

FIG. 1. Energy of positron resonances in the $\vec{\mathscr{E}}\parallel\vec{\mathscr{H}}$ fields at $\mathscr{H}=23.5$ kG. The solid curves are calculated using Eq. (4); the dashed curves give more accurate results of the summation of the $1/n$ expansion. The curves are labeled by the quantum numbers $|n_z, n_\rho, m\rangle$. The dot–dashed line shows the height of the potential barrier $E_s - E_0$.

energy (in powers of $\mathscr{E}^{1/4}$) when $\mathscr{E}\to 0$ and the ratio $\mathscr{H}/\mathscr{E}^3$ remains fixed.

The third term in (4) has a pole at $\alpha = 0$ corresponding to the disappearance of the minimum of the potential. Another pole, lying at $\alpha = \sqrt{2}/2$, corresponds to quasicrossing of several levels with the same sum $n_z + n_\rho$ and, hence, with the same oscillator energy. For the lowest levels $|0,0,m\rangle$ quasicrossing does not occur, and this pole, as expected, is canceled by the zero of the expression in square brackets. The sum and product of the energies of two excited states $|0,1,m\rangle$ and $|1,0,m\rangle$ also have no pole at $\alpha = \sqrt{2}/2$. Thus, it is possible to find these energies near the quasicrossing point by solution of a quadratic equation. Similarly, the energies of the states $|0,2,m\rangle$, $|1,1,m\rangle$, and $|2,0,m\rangle$ can be found by solving a cubic equation.

The results for the energies of different $m = 0$ levels, calculated by means of Eq. (1) and the above observations, are presented in Fig. 1. The magnetic field is fixed and amounts to 23.5 kG or 10^{-5} a.u. The disappearance of the minimum of the potential occurs at $\mathscr{E}_c = (\mathscr{H}/2)^{4/3} = 440$ V/cm, and the quasicrossing of the levels with the same values of $n_z + n_\rho$ occurs at $\mathscr{E}_q = \sqrt{2}/3\mathscr{E}_c = 359$ V/cm.

At $\mathscr{E} = 0$ there exist only the Landau levels with energies[2)] $N\mathscr{H}/2$. When $\mathscr{E} > 0$, owing to the possibility of quantization along the z axis, each Landau level becomes an infinite system of sublevels with different values of n_z. Quantum transitions between close quasistationary levels can be observed experimentally in the radio-frequency band.

The dashed curves in Fig. 1 show the exact energy obtained by summation of the $1/n$ expansion (see Sec. 4 below). It is clear that in order to describe the quasicrossing of the levels $|n_z,1,0\rangle$ and $|n_z+2,0,0\rangle$ higher terms of the expansion (4) must be included. With approach to the classical disintegration threshold E_s (the dot–dashed curve) the levels become wider, and with further growth of $\mathscr{E}$ they cease to be quasistationary.

When $\mathscr{E} > \mathscr{E}_c$ the potential no longer has a minimum, but analytic continuation of Eqs. (3) and (4) gives a complex energy which defines the position (Re E) and halfwidth ($-$ Im E) of the resonance. For instance, in a purely electric field, substituting $\alpha = -i$ into (4), we obtain

$$E = 2\mathscr{E}^{1/2} + \left(\frac{p}{\sqrt{2}} - iN\right)\mathscr{E}^{3/4} + \frac{1}{32}\left[3\left(N^2+m^2-\frac{p^2}{2}\right)-\frac{1}{2}\right]\mathscr{E} + \frac{1}{1024}\left[\frac{p}{\sqrt{2}}\left(\frac{23}{2}p^2+24N^2+24m^2-\frac{43}{2}\right) + iN(12p^2+23N^2+81m^2-31)\right]\mathscr{E}^{5/4} + O(\mathscr{E}^{3/2}), \quad (6)$$

where we have written out the term $\sim\mathscr{E}^{5/4}$ which is absent in (4).

3. Let us now choose the unperturbed potential V_0 and the perturbation W in a different way:

$$V_0 = {}^1/_8\mathscr{H}^2\rho^2 + 1/z + \mathscr{E}z,$$
$$W = V - V_0 = \frac{1}{r} - \frac{1}{z} = -\frac{1}{2}z^{-3}\rho^2 + \frac{3}{8}z^{-5}\rho^4 - \frac{5}{16}z^{-7}\rho^6 + O(\rho^8). \quad (7)$$

In the unperturbed Schrödinger equation the variables separate, and the wave function factorizes:

$$\chi(\rho, z) = F_{n_z}(z)\,|n_\rho, m\rangle, \quad E = E_\rho + E_z,$$

where $F_{n_z}(z)$ and E_z are the eigenfunction and eigenvalue of the one-dimensional Hamiltonian

$$H_z=-\frac{1}{2}\frac{d^2}{dz^2}+v(z),\quad v(z)=\frac{1}{z}+\mathscr{E}z,$$

and $|n_\rho,m\rangle$ and $E_\rho=N\mathscr{H}/2$ are the wave function and energy of the two-dimensional isotropic harmonic oscillator with the potential $\mathscr{H}^2\rho^2/8$. Consistent inclusion of the perturbing terms (7) leads to the following expansion for large values of $\mathscr{H}$:

$$E=\frac{N}{2}\mathscr{H}+E_z+\sum_{k=1}^{\infty}E^{(k)}\mathscr{H}^{-k}. \qquad (8)$$

The coefficients $E^{(1)}$, $E^{(2)}$, etc, are found by means of Rayleigh–Schrödinger perturbation theory. Making use of the fact that in the oscillator basis we have

$$\langle n_\rho,m|\rho^2|n_\rho,m\rangle=2N\mathscr{H}^{-1},$$
$$\langle n_\rho,m|\rho^2|n_\rho+2,m\rangle=[(N+m+1)(N-m+1)]^{1/2}\mathscr{H}^{-1},$$

we find that up to terms of the order $\mathscr{H}^{-3}$ the energy is determined by the one-dimensional Schrödinger equation

$$\{{}^1/_2N\mathscr{H}+H_z-Nz^{-3}\mathscr{H}^{-1}+{}^3/_4(3N^2-m^2+1)z^{-5}\mathscr{H}^{-2}$$
$$-[{}^5/_4N(5N^2-3m^2+7)z^{-7}+Nz^{-6}]\mathscr{H}^{-3}-E\}F(z)=0.$$

In order to obtain explicit formulas for the coefficients $E^{(k)}$ at small values of $\mathscr{E}$, we expand the potential $v(z)$ in the vicinity of the minimum $z_0=\mathscr{E}^{-1/2}$:

$$v(z)=2\mathscr{E}^{1/2}+(z-z_0)^2\mathscr{E}^{3/2}+\sum_{i=1}^{\infty}(z-z_0)^{i+2}\mathscr{E}^{(i+3)/2}.$$

Now, proceeding by analogy with Sec. 2, the energy E_z and the matrix elements entering into the formulas for $E^{(k)}$ can be expanded in powers of $\mathscr{E}^{1/4}$. As a result, we find

$$E^{(0)}=E_z=\mathscr{E}^{1/2}\left[2+\frac{\sqrt{2}}{2}p\mathscr{E}^{1/4}+\frac{1}{64}(-3p^2+5)\mathscr{E}^{1/2}\right.$$
$$\left.+\frac{\sqrt{2}p}{4096}(23p^2+5)\mathscr{E}^{3/4}+O(\mathscr{E})\right]$$

$$E^{(1)}=-N\mathscr{E}^{3/2}\left[1+\frac{3\sqrt{2}}{8}p+\frac{1}{64}(-3p^2+37)\mathscr{E}^{1/2}\right.$$
$$\left.+2^{-27/2}p(115p^2+8217)\mathscr{E}^{3/4}+O(\mathscr{E})\right],$$
$$(9)$$
$$E^{(2)}=\mathscr{E}^{5/2}\left\{3(1-m^2)\left[\frac{1}{4}+\frac{15\sqrt{2}}{32}p\mathscr{E}^{1/4}+\frac{1}{256}(75p^2+355)\mathscr{E}^{1/2}\right.\right.$$
$$\left.+2^{-31/2}p(175p^2+208845)\mathscr{E}^{3/4}\right]+N^2\left[\frac{27\sqrt{2}}{64}p\mathscr{E}^{1/4}\right.$$
$$\left.\left.+\frac{1}{256}(75p^2+611)\mathscr{E}^{1/2}+2^{-35/2}p(465p^2+892\,851)\mathscr{E}^{3/4}\right]+O(\mathscr{E})\right\},$$

$$E^{(3)}=-N\mathscr{E}^{3}\left[1+3\sqrt{2}\,p\mathscr{E}^{1/4}+\frac{1}{16}(57p^2+235)\mathscr{E}^{1/2}\right.$$
$$+\frac{\sqrt{2}p}{2048}(983p^2+110\,805)\mathscr{E}^{3/4}-N^2\left(\frac{1}{2}\mathscr{E}^{1/2}+\frac{425\sqrt{2}}{256}p\mathscr{E}^{3/4}\right)$$
$$\left.+15m^2\left(\frac{1}{8}\mathscr{E}^{1/2}+\frac{43\sqrt{2}}{128}p\mathscr{E}^{3/4}\right)+O(\mathscr{E})\right],$$

where N and ρ are given by (5).

Note that by substitution of (9) into (8) and regrouping it is easy to obtain the expansion of the energy for a fixed $\mathscr{H}$ and $\mathscr{E}\to 0$:

$$E={}^1/_2N\mathscr{H}+2\mathscr{E}^{1/2}+{}^1/_2\sqrt{2}p\mathscr{E}^{3/4}+\ldots$$

4. In the method of the $1/n$ expansion the energies of the states with fixed n_z, n_ρ and with $m\to\infty$ are expanded in powers of n^{-1}, where $n=n_z+n_\rho+m+1$:

$$E=n^{-2}\sum_{k=0}^{\infty}\varepsilon_k n^{-k}.$$

With growth of n the fields $\mathscr{E}$ and $\mathscr{H}$ decrease in such a way that the scaled intensities $F=n^4\mathscr{E}$ and $B=n^3\mathscr{H}$ remain constant. The details of the method are given in Ref. 3. As a rule, at first ($k\lesssim 3$) the coefficients ε_k decrease (in modulus) and then start to increase. In a sufficiently strong electric field $F>F_*(B)$ the coefficients ε_k become complex.

Summation of the $1/n$ expansion by the method of Padé approximants allows one to calculate the energy[3] even in the least favorable case $n=1$. The calculated values of $E(\mathscr{E},\mathscr{H})$ for the ground state are shown in Fig. 2 (in a.u.). With growth of $\mathscr{H}$ the level energy grows, whereas the width decreases. When $\mathscr{E}<\mathscr{E}_*=n^{-4}F_*$, the width of the level is very small, and the $1/n$ expansion does not allow one to determine it. For $\mathscr{H}=0$, 0.1, 0.2, 0.3, 0.4, and 0.5 we find $\mathscr{E}_*=0$, 0.010, 0.023, 0.037, 0.051, and 0.066, respectively.

5. When $\mathscr{H}=0$, the variables in the Schrödinger equation can be separated in the parabolic coordinates $\xi=r+z$ and $\eta=r-z$. The energy of the highly excited states is found by using the quasiclassical Bohr–Sommerfeld quantization conditions in the variables ξ and η.

When $m=0$, the required integrals can be calculated analytically as in the case of the Stark effect in the hydrogen atom.[4] As a result, we obtain the system of equations

$$\left.\begin{aligned}&\frac{E^{3/2}}{4\mathscr{E}}\left(1+\frac{4\beta_1\mathscr{E}}{E^2}\right)f\left(1+\frac{4\beta_1\mathscr{E}}{E^2}\right)=n_z+\frac{1}{2}\\&\frac{i\beta_2}{\sqrt{2}E^{1/2}}f\left(\frac{4\beta_2\mathscr{E}}{E^2}\right)=n_\rho+\frac{1}{2}\\&\beta_1+\beta_2=-1\end{aligned}\right\}, \qquad (10)$$

where $f(z)={}_2F_1(\frac{1}{4},\frac{3}{4},2,z)$ is the hypergeometric function. When $\mathscr{E}\to 0$, the quasiclassical energy can be expanded in powers of $\mathscr{E}^{1/4}$; the first two terms of the expansion coincide with the exact terms (6), and in the other terms only the coefficients of the leading powers of N and ρ are correct.

In the classical limit $n_z\to\infty$ the longest-living resonances $|n_z,0,0\rangle$ correspond to the oscillations of the positron along the z axis between the nucleus and the anode. The degree of overlap of the neighboring long-living resonances is characterized by the parameter $\Gamma/\Delta E$. When $\mathscr{E}\to 0$, we have $\Gamma/\Delta E=\sqrt{2}\simeq 1.41$. Numerical solution of the system (10) showed that with growth of $\mathscr{E}$ this ratio increases slightly, reaching a value 1.75 at $\mathscr{E}=20$ kV/cm for $n=30$. Therefore, as in the case $\mathscr{H}=0$, all the long-living resonances overlap each other, both in weak and in strong fields. In contrast to positron resonances, Stark electron resonances cannot overlap even at positive energies.[2]

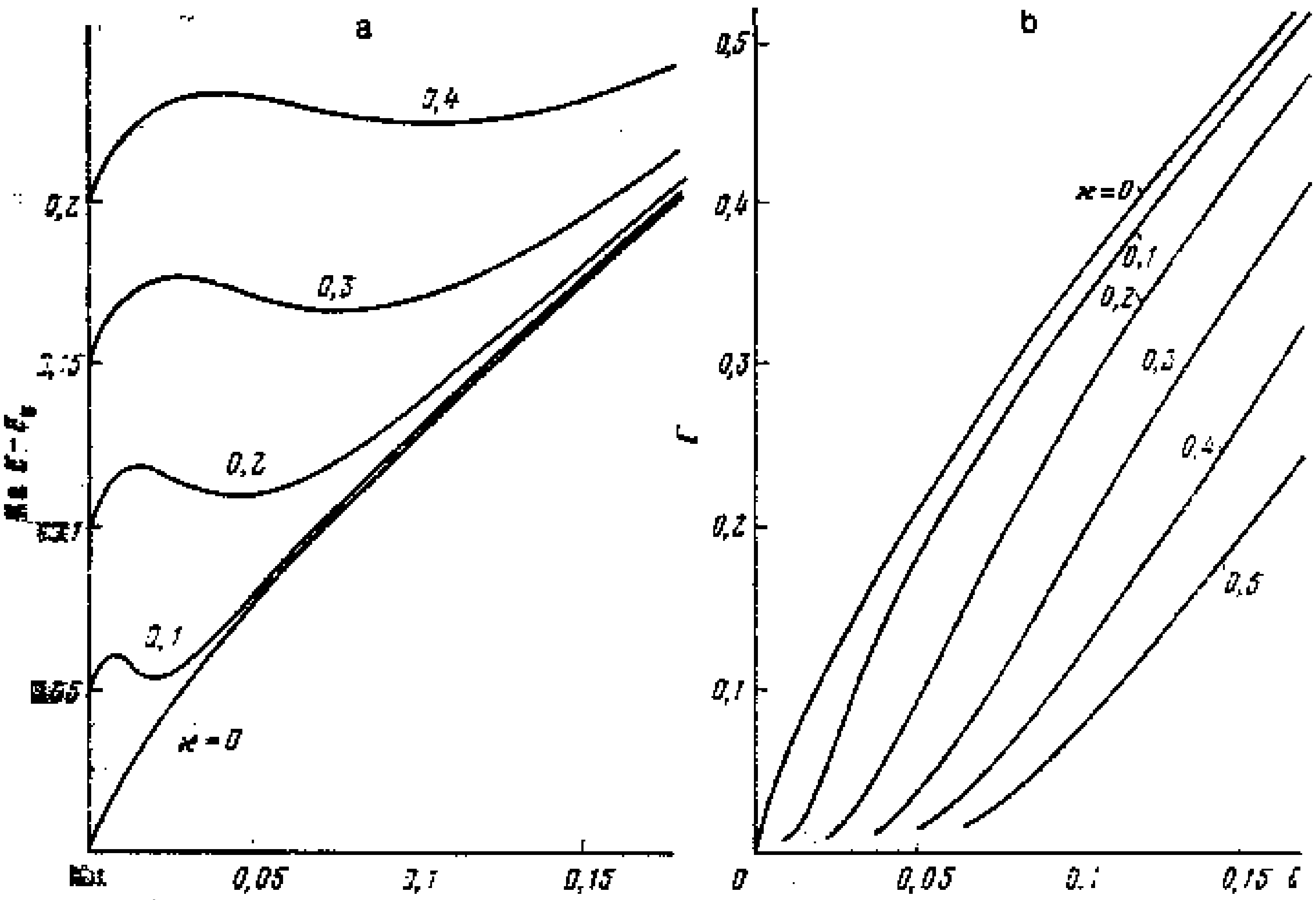

FIG. 2. Energy (a) and width (b) of the ground level as functions of $\mathscr{E}$ and $\mathscr{H}$. The numbers near the curves indicate the values of the magnetic field $\mathscr{H}$. The energy and the strengths of the fields $\mathscr{E}$ and $\mathscr{H}$ are measured in atomic units, and are, respectively, 27.21 eV, $5.142\cdot10^9$ V/cm, and $2.350\cdot10^9$ G.

6. Let us formulate our main results. When the condi-tion $\mathscr{H}^4 > 16\mathscr{E}^3$ is obeyed, positron resonances with ener-gies $E < E_t$ decay by quantum tunneling, and their width is [illegible] small. Depending on the choice of the unperturbed [illegible]miltonian, it is possible to expand the solutions both in a [illegible] electric field (4) and in a strong magnetic field (8). [illegible] energy can be found exactly by using the $1/n$ expansion; [illegible] case of large values of n_z and n_ρ and a purely electric [illegible] it is convenient to use the quasiclassical approximation. The effects of the finite nuclear mass, relativity, and the [illegible]ibility of annihilation have not been taken into account [illegible] require separate study.

Note that a pair of positrons or electrons does not form [illegible]ilar resonances, since by a transition to a moving refer-[illegible] frame the external electric field is eliminated.

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[1)] Positrons can be temporarily captured by an open resonator formed in the gap between the two nuclei.[1] Such resonances are not necessarily connected with the narrow positron lines observed in experiments (the so-called Darmstadt effect).

[2)] In contrast to the hydrogen atom, in the limit $\mathscr{E}\to0$ the pe^+ system disintegrates (the positron moves away from the nucleus and goes onto a Landau level); therefore, the energy and other quantities have a singularity at $\mathscr{E}=0$. For instance, the distance between neighboring levels is $\Delta E\sim\mathscr{E}^{3/4}$. The same unusual behavior of ΔE at $\mathscr{E}\to0$ is also characteristic of Stark near-threshold resonances, where it is a consequence of the quasiclassical limit.[2]

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