Physics of Atomic Nuclei, Vol. 57, No.6, 1994, pp. 1057 - 1065. Translated from Yadernaya Fizika, Vol. 57, No. 6, 1994, pp. 1119 - 1128.

ELEMENTARY PARTICLES AND FIELDS
Theory

On the Asymptotics of High-Order Terms of the 1/n Expansion

V. S. Popov, A. V. Sergeev[1)], V. D. Mur[2)], and A. V. Shcheblykin
Institute of Theoretical and Experimental Physics, Moscow, Russia
Received July 12, 1993

Abstract – Analytic and numerical methods for determining the asymptotics of high-order terms of the 1/n expansion in quantum-mechanical problems are developed. It is shown that this asymptotics is always of the factorial type. The dependence of parameters of the asymptotics on the form of the potential and on the coupling constant is especially analyzed in the vicinity of the point of collision of classical solutions.

1. The $1/n$ expansion is a new and promising method in quantum mechanics (see, for example, [1 - 3]). The behavior of high-order terms of the $1/n$ expansion and the convergence of corresponding series have been recently studied using numerical [4, 5], as well as analytic [6, 7], methods. In this paper, we continue to study these problems, paying special attention to short-range potentials encountered in quantum mechanics and in atomic and nuclear physics.

There are several versions of this method differing in the choice of the expansion parameter (see [1, 2] for details). We will consider below a version of the $1/n$ expansion proposed in [8]. The peculiar feature of this approach is that it is possible to apply it not only to the levels of a discrete spectrum, but also to quasi-stationary states (resonances). The corresponding examples can be found in [8 - 10].

2. Let us briefly describe the main aspects of the method for determining the asymptotics of high-order terms of the $1/n$ expansion. The eigenvalues of energy, which are complex in the case of quasi-stationary states ($E = E_r - i\Gamma/2$), can be written in the form of power series in the "small parameter" $1/n$:

$$\varepsilon_{nl} \equiv 2n^2 E_{nl} = \varepsilon^{(0)} + \frac{\varepsilon^{(1)}}{n} + \ldots + \frac{\varepsilon^{(k)}}{n^k} + \ldots, \qquad (1)$$

where $n = n_r + l + 1$ is the principal quantum number of the level ($n_r = 0, 1, \ldots$ is fixed, while the orbital momentum $l \longrightarrow \infty$), k is the order of the $1/n$ expansion, and ε_{nl} is the "reduced" energy. The coefficients $\varepsilon^{(k)}$, which will be referred to as high-order terms of the $1/n$ expansion, can be calculated using recurrence relations [4]. After the calculation, the question about how to sum series (1) arises. Here, we need information on the asymptotic behavior of the coefficients $\varepsilon^{(k)}$ for $k \longrightarrow \infty$.

In the cases under investigation, this asymptotics is of the factorial type:

$$\varepsilon^k = k!\,a^k k^\beta \left(c_0 + \frac{c_1}{k} + \frac{c_2}{k^2} + \ldots\right), \quad k \longrightarrow \infty, \qquad (2)$$

where a, β, c_0, etc. are the calculatable parameters. The asymptotics of this type is well known for series of ordinary perturbation theory (in powers of the coupling constant g) in quantum mechanics and field theory, where it is related to the instability of the vacuum state under the sign reversal of g.[3)] It will be shown later that, in some cases, the asymptotics is more complicated, e.g.,

$$\varepsilon^{(k)} = k!\,\{a^k k^\beta c_0 [1 + O(1/k)] + 2\mathrm{Re}(a_c^k k^{\beta_c} c_c) + \ldots\}, \qquad (3)$$

where a_c and c_c are complex parameters depending on g. The role of two terms in (3) may change as a result of a change in g.

The two terms dominating in the asymptotic expansion of $\varepsilon^{(k)}$ can be determined from the behavior of the signs of the coefficients of $\varepsilon^{(k)}$: if series (1) is a series of terms with constant or alternating signs, the first term of the "Dyson" type is dominant (in this case, the parameter $a > 0$ or $a < 0$, respectively). If, however, the signs of $\varepsilon^{(k)}$ with large numbers k behave irregularly (i.e., oscillations of high-order terms are observed in the $1/n$ expansion), the second term in expression (3) obviously plays an important role. This simple consideration indicates that the parameters (a, a_c, β, β_c, etc.) of the asymptotics can be determined if the coefficients of the $1/n$ expansion are found numerically up to fairly high orders k.[4)]

Let us discuss several theoretical formulas. The parameter a plays the main role (after $k!$) in the asymp-

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[1)] Vavilov State Optical Institute, St. Petersburg, Russia

[2)] Moscow Institute of Engineering Physics, Kashirskoe sh. 31, Moscow 115409, Russia

[3)] This is the so-called "Dyson phenomenon", established for the first time in quantum electrodynamics [11], and then, observed in many problems of quantum mechanics [12 - 16] and field theory [17].

[4)] In our calculations, the order k actually did not exceed 40 (the calculations were made with a "quadruple accuracy", i.e., to within 32 decimal places in the coefficients $\varepsilon^{(k)}$). An advancement to higher orders is not fraught with difficulties of principle nature; however, we encounter technical problems associated with the accumulation of errors in the calculation of the coefficients $\varepsilon^{(k)}$ by recurrence relations. We will consider further details of computational nature in Appendix 1.

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totic expansions (2) and (3). For potentials with a spherical symmetry, it is defined by the formula

$$a = (2I_0)^{-1}, \quad I_0 = \int_{r_0}^{r_2} q(r)dr, \qquad (4)$$

where $q = [2U(r) - \varepsilon^{(0)}]^{1/2}$, $U(r)$ is the effective potential including the centrifugal energy, i.e.,

$$U(r) = \left(2r^2\right)^{-1} + \lim_{n \to \infty} n^2 V(n^2 r),$$

and $V(r)$ is the initial potential appearing in the Schrödinger equation.[5)] For example, for screened Coulomb potentials

$$V(r) = -r^{-1}f(\mu r), \quad \hbar = m = e = 1, \qquad (5)$$

we have $U(r) = (2r^2)^{-1} - r^{-1}f(\nu r)$, where we introduced the parameter $\nu = n^2\mu$, which is convenient for states with $n \gg 1$ [here, $f(x)$ is the screening function; see, for example, formulas (8) and (12)]. As $n \longrightarrow \infty$, a particle is localized in the neighborhood of the classical point of equilibrium $x = x_0(\nu)$ (here, $x = n^2\mu r$) in the effective potential, which can be determined from the equation [8]

$$xf - x^2f' = \nu, \qquad (6)$$

and the $1/n$ expansion is constructed around the point x_0. The reduced energy $\varepsilon^{(0)}$ corresponds to the minimum of the potential $U(r)$, and hence, two turning points coincide: $r_0 = r_1$, i.e., the classically allowed region shrinks to a point as $n \longrightarrow \infty$, and the domain $r_0 < r < r_2$ is the subbarrier region in which $q^2 > 0$.

It can be proved [6] that, for states without radial nodes ($n_r = 0$), we have $\beta = -3/2$. The preexponential factor c_0 can be calculated by the formula

$$c_0 = -\left(\frac{\omega}{\pi}\right)^{3/2} a^{-1/2} (r_2 - r_0) \exp(\omega I_1 + I_2), \qquad (7)$$

which can be derived in the same way as formula (4) (see also [18, 19]). Here,

$$I_1 = \int_{r_0}^{r_2} \left[q^{-1} - \frac{1}{\omega(r - r_0)}\right]dr,$$
$$I_2 = \int_{r_0}^{r_2} \left(\frac{1}{r^2} - \frac{1}{r_0^2}\right)\frac{dr}{q(r)}, \qquad (7a)$$

and $\omega = [U''(r_0)]^{1/2}$ is the frequency of classical oscillations about the point of equilibrium r_0.

Thus, the parameters appearing in the asymptotics of high-order terms of the $1/n$ expansion are defined analytically. Let us consider some examples.

[5)]See Fig. 3 in [6] for notation (in this figure, $1/2\varepsilon^{(0)}$ must be plotted instead of $\varepsilon^{(0)}$ on the ordinate axis). If there are several turning points $r_2^{(i)}$, we must choose the one for which the modulus of the corresponding integral is minimum.

3. We will consider the results obtained for short-range potentials. We begin with the Yukawa potential

$$V(r) = -\frac{g}{r}\exp(-r/R), \qquad (8)$$

which can be reduced to the form (5) by means of the gauge transformation $r \longrightarrow \alpha r$ in the Schrödinger equation. For $\alpha = g^{-1}$, we obtain

$$f(x) = e^{-x}, \quad x = \mu r, \quad \mu = (gR)^{-1}, \quad E_n = g^2\tilde{E}_n, \qquad (8a)$$

where $\tilde{E}_n$ is the eigenvalue of energy for potential (5). In this case, the value $\nu_{cr} = n^2\mu_{cr} = 0.73576...$ corresponds [8] to the passage of the highly excited level with $n = l + 1 \gg 1$ to the continuous spectrum, while $\nu_* = 0.83996...$ corresponds to merging of two points of equilibrium (stable and unstable) in the effective potential $U(r)$, after which the latter has no minimum for real $0 < r < \infty$.

As $\nu \longrightarrow 0$, potential (8) is transformed to the Coulomb potential, and $a(\nu) \approx \nu/2ln\nu \longrightarrow 0$. Numerical analysis shows that, for $\nu < \nu_{cr}$, the coefficients $\varepsilon^{(k)}$ oscillate for large values of k:

$$\varepsilon^{(k)} \sim k!|a_c|^k\cos(k\varphi + \varphi_0), \quad \varphi = \arg a_c. \qquad (9)$$

For $\nu = 0.5257$, we have $\mathrm{Re}\,a_c = 0$ and $\varphi = \pi/2$; therefore, the period of sign reversal for $\varepsilon^{(k)}$ is equal to four (see Table 1).

For $\nu_{cr} < \nu < \nu_*$, the first term in expression (3) contains $a(\nu) > 0$, and the series becomes a series of constant sign (for example, for $\nu = 0.8$, the coefficients $\varepsilon^{(k)} < 0$ for $k \geq 4$). For $\nu \longrightarrow \nu_*$, the parameter of the asymptotics has a power singularity:

$$a(\nu) = A(1 - \nu/\nu_*)^{-5/4}[1 + O((\nu_* - \nu)^{1/2})], \qquad (10)$$

where $A = 0.1116$ (the exponent $-5/4$ does not depend on the specific form of the potential [6]). Consequently, the rate of growth of coefficients $\varepsilon^{(k)}$ increases sharply as ν approaches ν_*. This is clearly demonstrated in Table 1.

An analytic continuation of (10) to the region $\nu > \nu_*$ leads to complex values of $a(\nu)$. This is in agreement with the fact that the coefficients of the $1/n$ expansion become complex-valued in this region [8]. This makes it possible to describe (by simple summation of the few first terms of the $1/n$ expansion) not only the shift, but also the width of a quasi-stationary level [8 - 10].

The results of the numerical calculation of high-order terms of the $1/n$ expansion are presented in Fig. 1, which gives the values of b_k, defined as

$$b_k = \ln(|\varepsilon^{(k)}(\nu)|/k!). \qquad (11)$$

If asymptotic expansion (3) is dominated by the first term, the dependence of b_k on k must approach a linear function with increasing k. This is actually the case for $\nu > \nu_{cr}$. In the neighborhood of the point ν_{cr}, the changeover of the regime takes place: for $\nu < \nu_{cr}$, the second term in expansion (3) becomes dominant; as a result, the growth of $|\varepsilon^{(k)}|$ becomes less regular, and the quantities b_k develop oscillations (see the curves for $\nu = 0.5$ and 0.7).

The same conclusion also follows from the calculation of the parameters a and a_c (see Fig. 2, which shows that the type of the singularity of (10) is completely confirmed).[6)]

Another potential widely used in quantum mechanics and the theory of the nucleus is the Hulthén potential $V(r) = V_0\{\exp(r/R) - 1\}^{-1}$ for which

$$f(x) = x/(e^x - 1), \quad \mu = (V_0R^2)^{-1}, \tag{12}$$

the equation for $x_0(\nu)$ has the form

$$x^3e^x(e^x - 1)^{-2} = \nu, \tag{13}$$

and the values of the characteristic parameters are as follows: $\nu_{cr} = 1.2952$, $\nu_* = 1.5234$, and $A = 0.1371$. The quantities a and a_c, as functions of ν, are presented in Fig. 3.

4. We consider short-range potentials of a more general form.

(a) Let the screening function in expression (5) be given by

$$f(x) = x^{\lambda-1}\exp(-x), \quad \lambda > 0 \tag{14}$$

($\lambda = 1$ corresponds to the Yukawa potential, $\lambda = 2$ corresponds to the exponential potential, and the condition $\lambda > 0$ rules out the "falling to the center" [20]). The point of equilibrium is determined from the equation

$$[x^{\lambda+1} + (2-\lambda)x^{\lambda}]\exp(-x) = \nu, \tag{15}$$

which has two positive roots. We must choose the smaller root $x = x_0(\nu)$ because $\omega^2 > 0$ for it:

$$\omega = \left\{\left(\lambda - \frac{x_0(1-\lambda+x_0)}{2-\lambda+x_0}\right)\right\}^{1/2} \sim \nu^{1/\lambda} \text{ for } \nu \longrightarrow 0$$

(ω is the dimensionless frequency of oscillations about the equilibrium point). Simple calculations yield

$$\nu_{cr} = 2(e^{-1}\lambda)^{\lambda}, \quad e = 2.71828\ldots, \tag{16}$$

$$\nu_* = \frac{1}{2\lambda}[1 + (1+4\lambda)^{1/2}]x_*^{\lambda+1}\exp(-x_*), \tag{17}$$

where $x_* = \lambda + \frac{1}{2}[(1+4\lambda)^{1/2} - 1]$ is the value of $x_0(\nu)$ corresponding to the merging of the roots (i.e., $\nu = \nu_*$). For the coefficient of the power singularity (10), we obtain

$$A = \frac{5}{96}(1+4\lambda)^{3/8}[(1+4\lambda)^{1/2} - 1]^{3/4}. \tag{18}$$

The dependences of ν_{cr}, ν_*, and A on λ are presented in Figs. 4 and 5. In particular, we have

$$A = 5^{11/8}(\sqrt{5}-1)^{3/4}/96 = 0.111646\ldots$$

for the Yukawa potential and

$$\nu_{cr} = 8e^{-2} = 1.0827, \quad \nu_* = 27e^{-3} = 1.3442,$$

and $A = 0.1997$ for the exponential potential.

[6)]The methods of calculation of these parameters from the coefficients $\varepsilon^{(k)}$ of the $1/n$ expansion are described in Appendix 1.

Table 1. Coefficients of the $1/n$ expansion for the Yukawa potential

k	$\varepsilon^{(k)}$		
	$\nu = 0.40$	$\nu = 0.5257$	$\nu = 0.8$
0	−3.449(−1)	−1.940(−1)	4.280(−2)
1	5.789(−2)	−9.193(−2)	−1.831(−1)
2	6.598(−3)	1.449(−2)	6.592(−2)
3	−5.205(−4)	−1.381(−3)	1.478(−2)
4	−6.019(−5)	−4.868(−4)	−2.940(−2)
5	2.749(−5)	1.822(−4)	−4.879(−1)
6	2.930(−6)	1.253(−4)	−7.608
7	−4.267(−6)	−7.772(−5)	−1.387(2)
8	2.164(−8)	−7.293(−5)	−2.902(3)
9	1.153(−6)	6.209(−5)	−6.869(4)
10	−2.300(−7)	7.320(−5)	−1.817(6)
11	−4.504(−7)	−7.818(−5)	−5.325(7)
12	2.303(−7)	−1.118(−4)	−1.713(9)
13	2.246(−7)	1.423(−4)	−6.008(10)
14	−2.375(−7)	2.411(−4)	−2.282(12)
15	−1.218(−7)	−3.546(−4)	−9.335(13)
16	2.807(−7)	−6.987(−4)	−4.093(15)
17	3.880(−8)	1.162(−3)	−1.915(17)
18	−3.823(−7)	2.620(−3)	−9.526(18)
19	9.997(−8)	−4.850(−3)	−5.020(20)
20	5.870(−7)	−1.235(−2)	−2.793(22)
21	–	2.517(−2)	−1.637(24)
22	–	7.161(−2)	−1.008(26)

Note: The order of a number is given in parentheses, e.g., −3.449(−1) = −0.3449.

(b) Let us consider another example:

$$f(x) = \exp(-\beta^{-1}x^{\beta}). \tag{19}$$

In this case, instead of (15), we obtain

$$(x_0 + x_0^{\beta+1})f(x_0) = \nu, \quad \omega = \left[\frac{1 + \beta x_0^{\beta} - x_0^{2\beta}}{1 + x_0^{\beta}}\right]^{1/2}. \tag{20}$$

It follows from these relations that

$$\nu_{cr} = 2\exp(-\beta^{-1}), \tag{21}$$

$$x_* = \left[\left(1 + \frac{\beta^2}{4}\right)^{1/2} + \frac{\beta}{2}\right]^{1/\beta}, \tag{22}$$

and ν_* is determined in terms of x_* according to the first equation from (20). Accordingly, for the constant A in expression (10), we obtain

$$A = \frac{5}{96}\left[\beta^2 + 4 + (\beta - 2)(\beta^2+4)^{1/2}\right]^{3/4}. \tag{23}$$

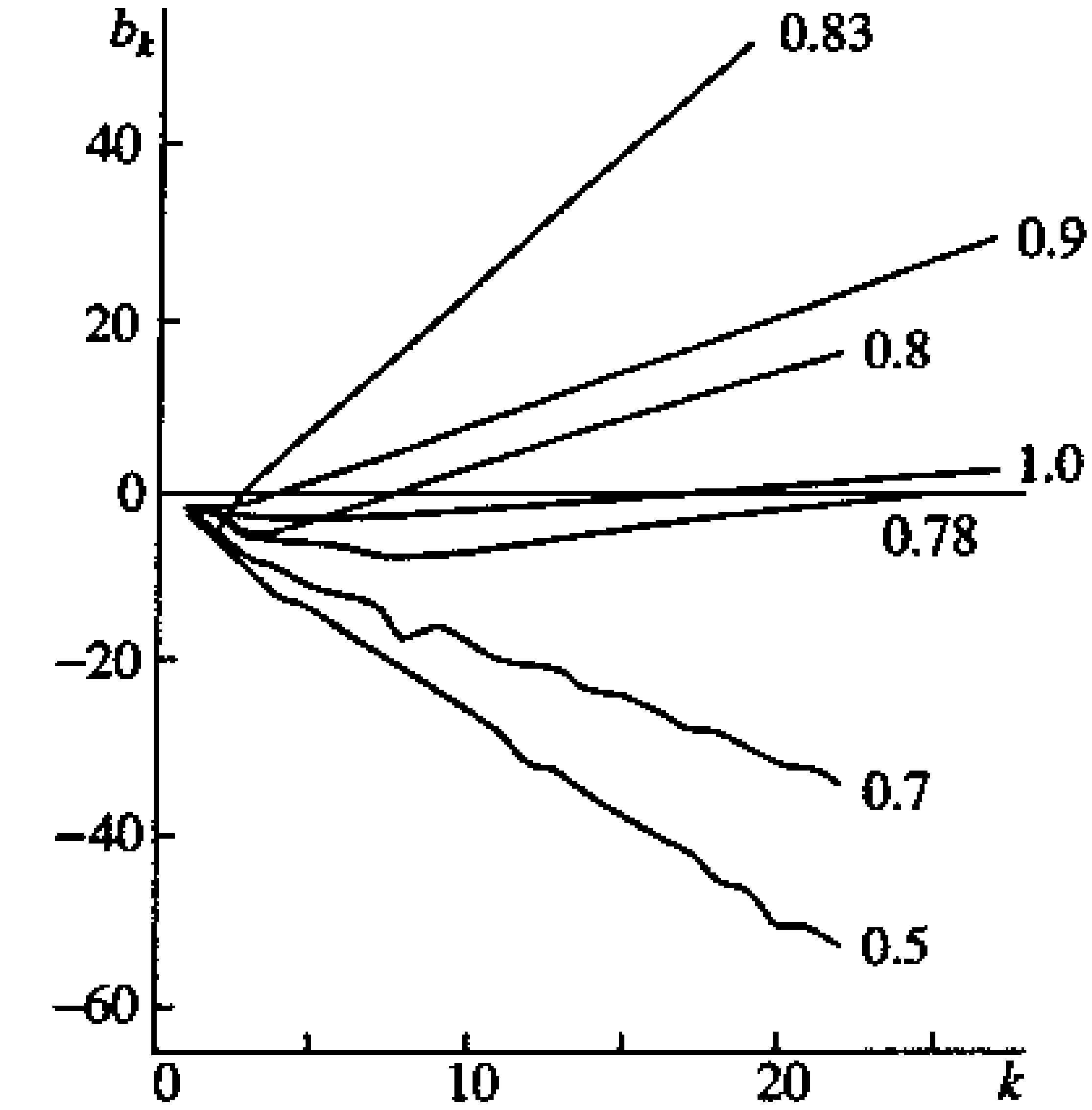

Fig. 1. The approach of the quantities b_k to the asymptotic values [see formula (11)] in the case of the Yukawa potential; k is the order of the $1/n$ expansion. The values of ν are indicated on the curves.

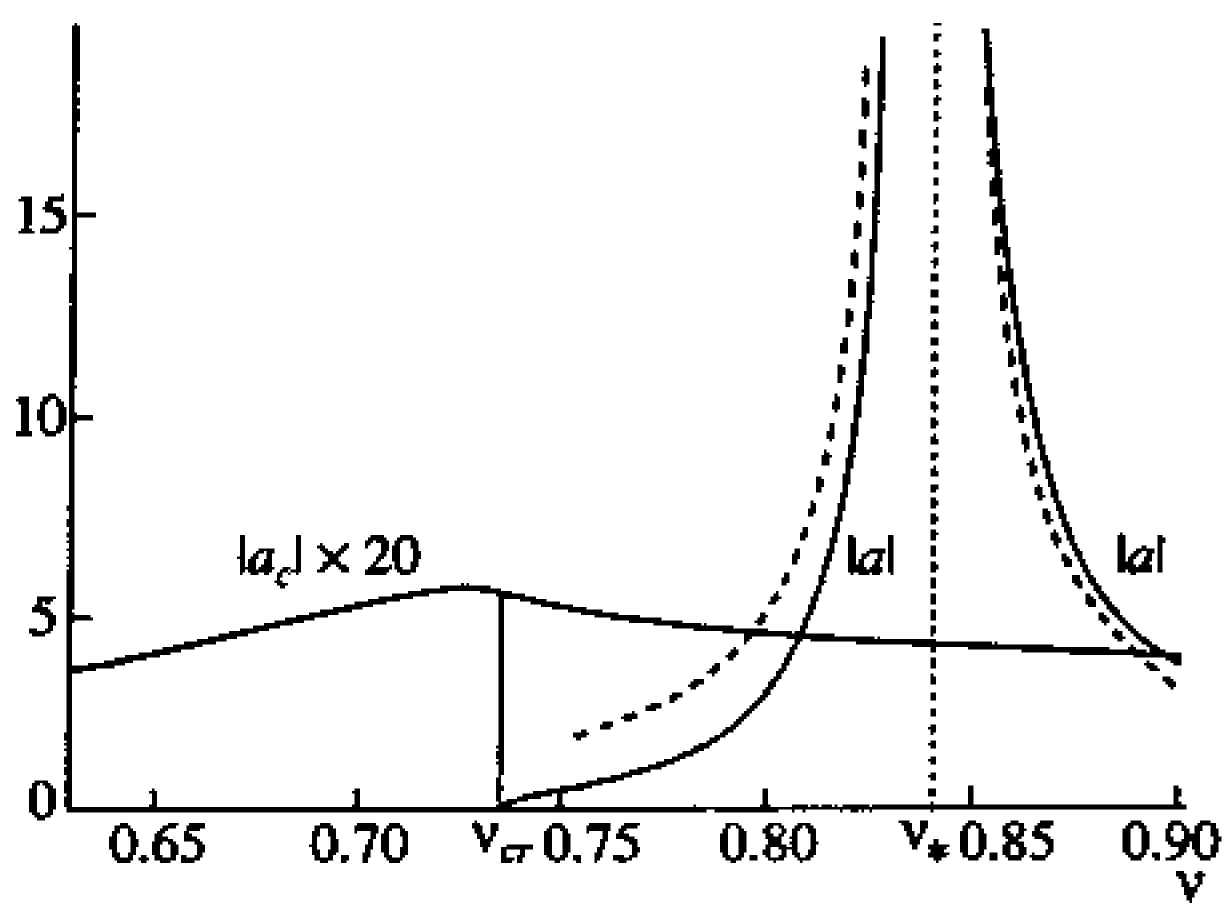

Fig. 2. The parameters a and a_c of the asymptotics for the Yukawa potential (the values of $|a_c|$ are magnified by a factor of 20). The dashed curve corresponds to formula (10). For $\nu_{cr} < \nu < \nu_*$, the parameter $a(\nu) > 0$.

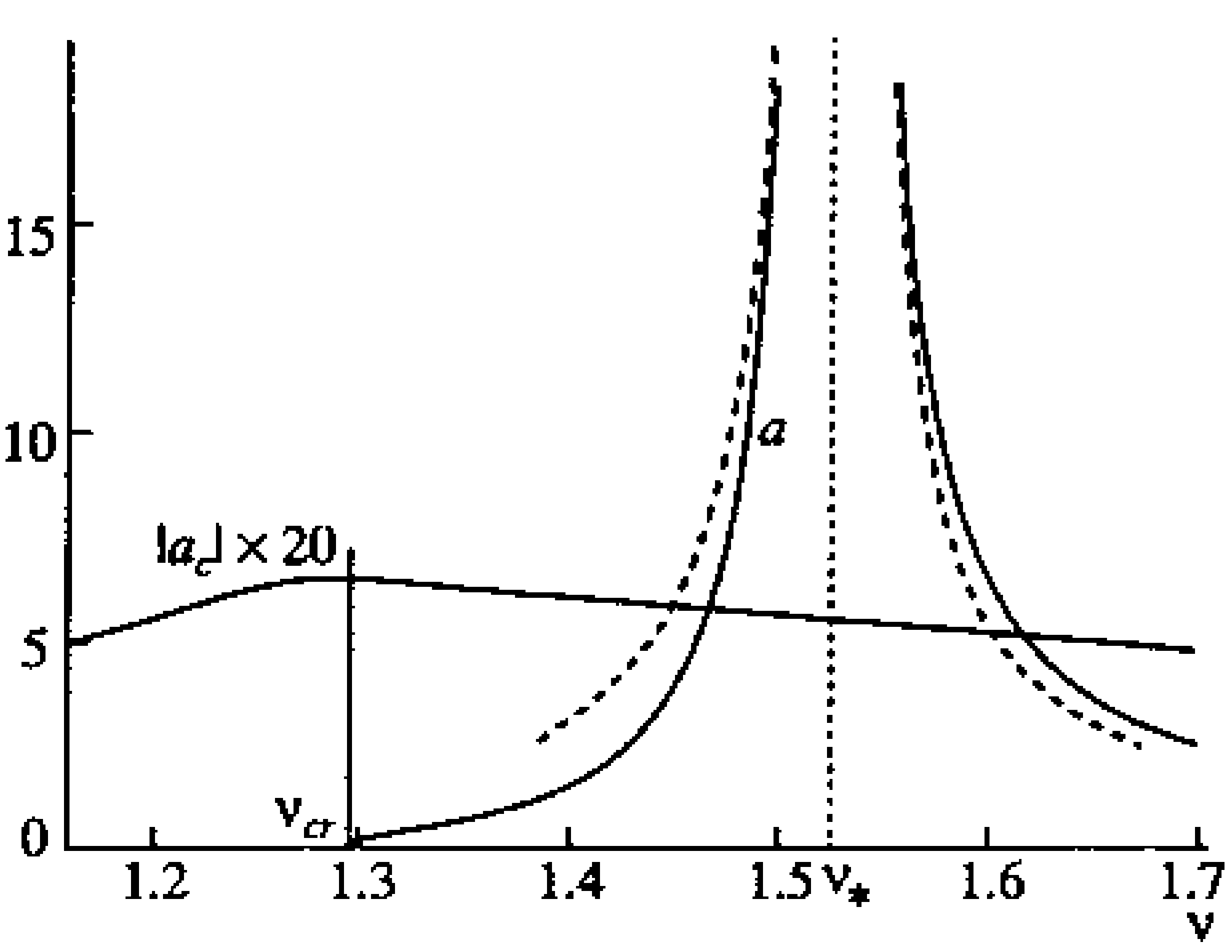

Fig. 3. The same as in Fig. 2 for the Hulthén potential.

The results of calculations are presented in Figs. 5 and 6 (the value $\beta = 1$ corresponds to the Yukawa potential and $\beta = 2$ corresponds to the Gaussian screening).

(c) Power-type screening:

$$f(x) = (1 + x/\kappa)^{-\kappa}, \quad \kappa > 1 \tag{24}$$

($\kappa = 2$ corresponds to the Tietz potential [21], which is well known from atomic physics and which is a good approximation of the Thomas–Fermi potential in neutral atoms, while $\kappa = \infty$ corresponds to the Yukawa potential; the condition $\kappa > 1$ is required in order for potential (5) to decrease at infinity more rapidly than the centrifugal potential). In this case, we have

$$[x_0 + (1 + \kappa^{-1})x_0^2]\,(1 + x_0/\kappa)^{-(\kappa+1)} = \nu, \tag{25}$$

$$\nu_{cr} = 2(1 - \kappa^{-1})^{\kappa-1}, \tag{26}$$

$$\nu_* = x_*^3\,(1 + \kappa^{-1})\,(1 + \kappa^{-1}x_*)^{-(\kappa+2)}, \tag{27}$$

where $\left[\left(\frac{5}{4} + \kappa^{-1}\right)^{1/2} - \left(\frac{1}{2} + \kappa^{-1}\right)\right]^{-1}$.

The ratio $\rho = (\nu_* - \nu_{cr})/\nu_*$ as a function of $1/\kappa$ is depicted in Fig. 7 (in particular, $\nu_{cr} = 1$, $\nu_* = 1.0563$, and $\rho = 0.0533$ for the Tietz potential). The coefficient A in expression (10) increases monotonically with κ, varying from $A = 0.05742$ at $\kappa = 2$ to $A = 0.11165$ at $\kappa = \infty$.

The obtained results indicate that the dependence of the asymptotic parameter a on ν is qualitatively the same for all of the potentials considered above.

It should be noted that the interval $\nu_{cr} < \nu < \nu_*$ is unfavorable for the $1/n$ expansion: for $\nu > \nu_{cr}$, the bound levels become quasi-stationary, while the classical point of equilibrium $x_0(\nu)$ has not yet passed to the complex plane, and hence, the coefficients in $\varepsilon^{(k)}$ are still real-valued. For this reason, simple summation of series (1) does not allow us to determine, in this case, the level width (see [8 - 10]).[7)] It can be seen from Figs. 4, 6, and 7 that the values of ν_{cr} and ν_* are numerically close in cases that are interesting from the physical point of view, and hence, the $1/n$ expansion is inapplicable only in a comparatively narrow region $\nu \approx \nu_*$ (this conclusion was drawn earlier [8] for the special case of the Yukawa potential). The width of this region can be estimated by using the parameter ρ defined as

$$\rho = (\nu_* - \nu_{cr})/\nu_* \tag{28}$$

($\rho = 0.1241$ and 0.1498 for the Yukawa and Hulthén potentials, respectively). It follows from the examples considered above that $\rho \longrightarrow 0$ in the cases when the potential $V(r) \propto r^{-2}$ for $r \longrightarrow 0$ or for $r \longrightarrow \infty$. For example, $\rho = c_1\lambda + O(\lambda^2)$ for $\lambda \longrightarrow 0$ in the case of screening (14), while for the screening (24), we obtain

[7)]This drawback can be overcome by using more powerful methods of summation of divergent series (e.g., with the help of the Padé–Hermite approximants).

$\rho = c_1(\kappa - 1) + \ldots$, $\kappa \longrightarrow 1$. The coefficients c_1 and c_1' are numerically small, i.e.,

$$c_1 = \ln 2 - 1/2 = 0.1931,$$
$$c_1' = \ln(2/3) + 1/2 = 0.0945. \tag{29}$$

This is in agreement with Figs. 4 and 7.

5. Let us consider an example of a potential for which the dependence of the asymptotic parameter a on ν has a form that differs qualitatively from those in Figs. 2 and 3. We consider the (generalized) funnel potential

$$V(r) = -r^{-1} + \frac{g}{N}r^N \quad (N, g > 0), \tag{30}$$

which can be written in the form (5) by setting

$$\nu = n^2 g^{1/(N+1)}, \quad f(x) = 1 - N^{-1}x^{N+1}$$

($N = 1$ corresponds to the funnel potential, which is often used in QCD). The corresponding effective potential $U(r)$ has a minimum for all $0 < g < \infty$. The classical point of equilibrium can be determined from the equation $x^{N+2} + x = \nu$ from which we obtain

$$x_0(\nu) = \begin{cases} \nu\,[1 - \nu^{N+1} + O(\nu^{2N+2})], & \nu \longrightarrow 0 \\ \nu^{\frac{1}{N+2}}\left(1 - \frac{1}{N+1}\nu^{-\frac{N+1}{N+2}} + \ldots\right), & \nu \longrightarrow \infty. \end{cases} \tag{31}$$

The frequency of oscillations about the point x_0 does not vanish:

$$\omega = \left[\frac{1 + (N+2)\,x_0^{N+1}}{1 + x_0^{N+1}}\right]^{1/2}$$
$$= \begin{cases} 1 + \frac{1}{2}(N+1)\,\nu^{N+1} + \ldots, & \nu \longrightarrow 0 \\ (N+2)^{1/2} - c_2\nu^{-\frac{N+1}{N+2}} + \ldots, & \nu \longrightarrow \infty \end{cases} \tag{32}$$

($c_2 > 0$), and hence, classical solutions do not merge in this case. Accordingly, the parameter $a(\nu)$ is complex-valued and does not have singularities of the type (10) (see Fig. 8). In light of the results obtained in Sections 3 and 4, this is not surprising because potential (30) has only a discrete spectrum, and the energy levels do not become quasi-stationary.

If $\nu \longrightarrow 0$, the potential in question approaches the Coulomb potential, and we have [see (A.14)]

$$a(\nu) \sim \nu^{\frac{N+1}{N}} \longrightarrow 0 \tag{33}$$

(in the case of the Coulomb potential, the $1/n$ expansion (1) is truncated because $\varepsilon^{(k)} \equiv 0$ for $k \geq 1$). In the limit $\nu \longrightarrow \infty$, the parameter $a(\nu)$ approaches a constant value that is determined only by N (see Table 2).

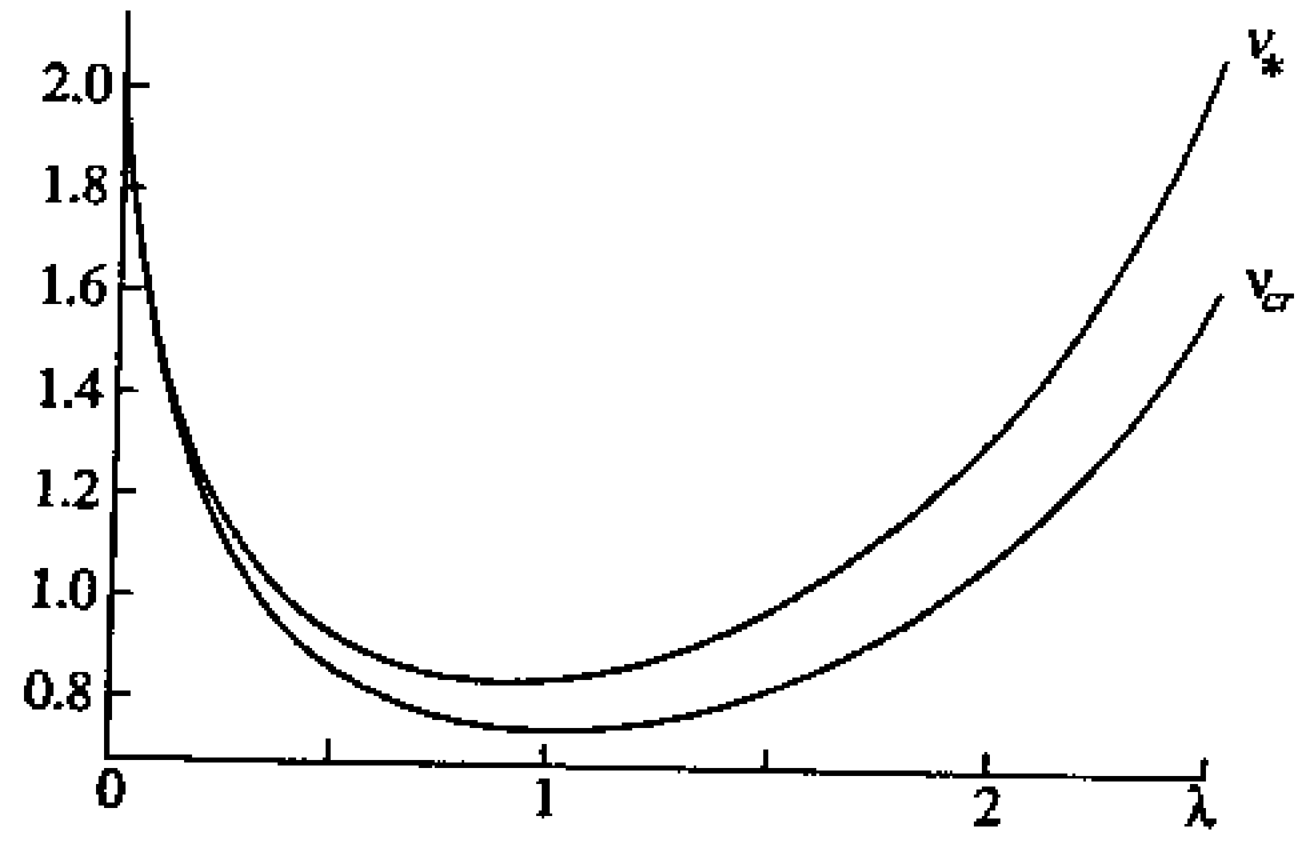

Fig. 4. The quantities ν_{cr} and ν_* for potentials (14). The minimum value of ν_{cr} is attained for the Yukawa potential ($\lambda = 1$).

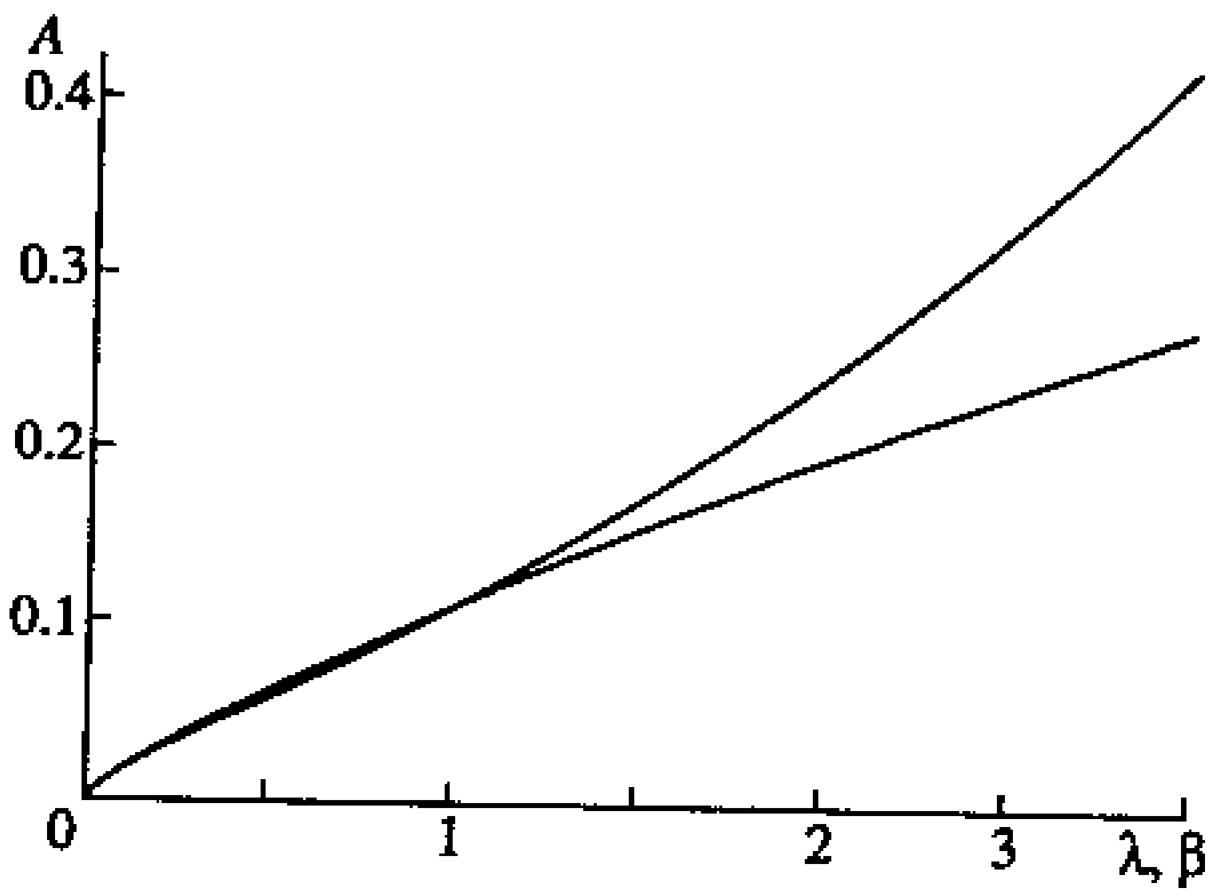

Fig. 5. The coefficient A of the power singularity (10). Curve 1 (dependence on λ) and curve 2 (dependence on β) correspond to potentials (14) and (19), respectively.

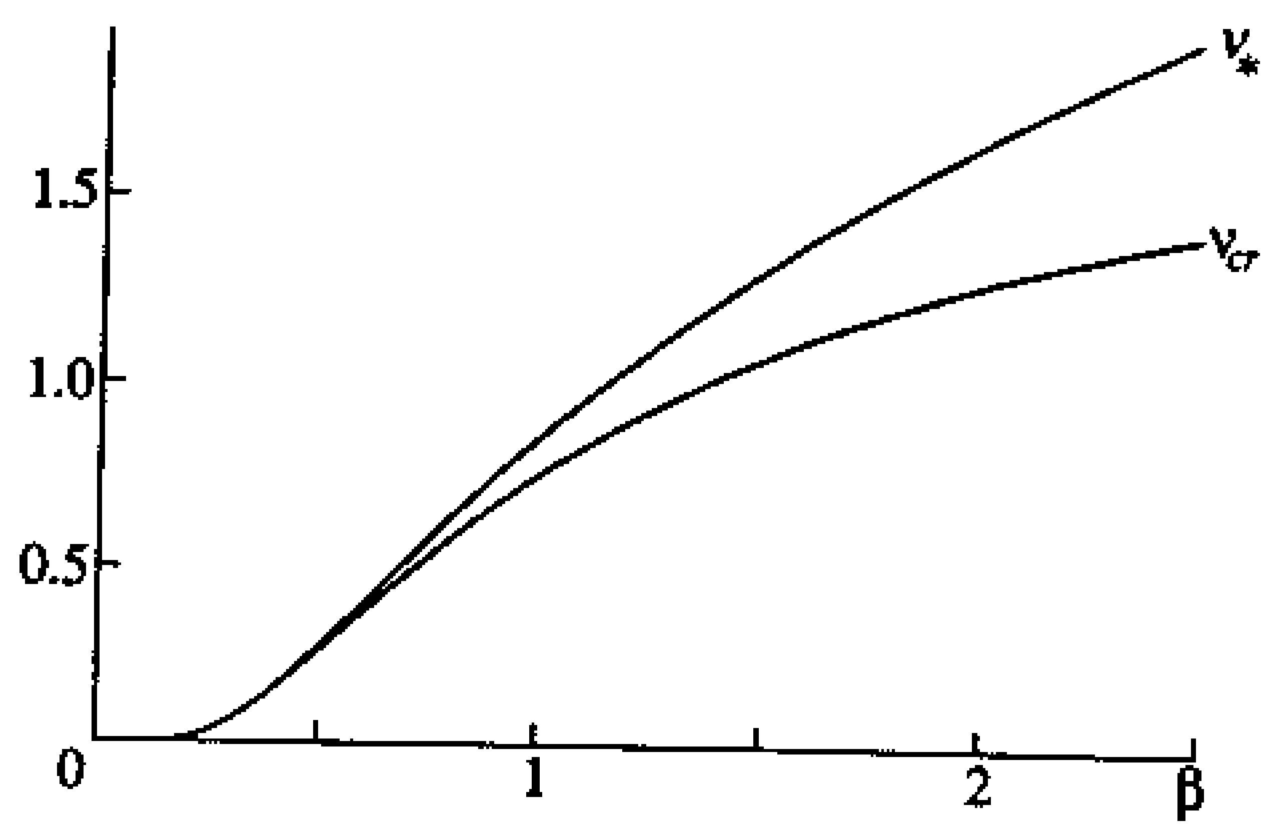

Fig. 6. The quantities ν_{cr} and ν_* for potentials (19).

Using formula (4), we can determine $a(\nu)$ in two limiting cases: $\nu \longrightarrow 0$ and $\nu \longrightarrow \infty$ (see Table 2 and Appendix 2).

6. Finally, let us consider the following question. It is well known that the coefficients E_k in the series

$$E(g) = \sum_k E_k g^k \tag{34}$$

of ordinary perturbation theory in powers of the coupling constant g may increase as $E_k \sim (k\alpha)!\, a^k$ with an arbitrary

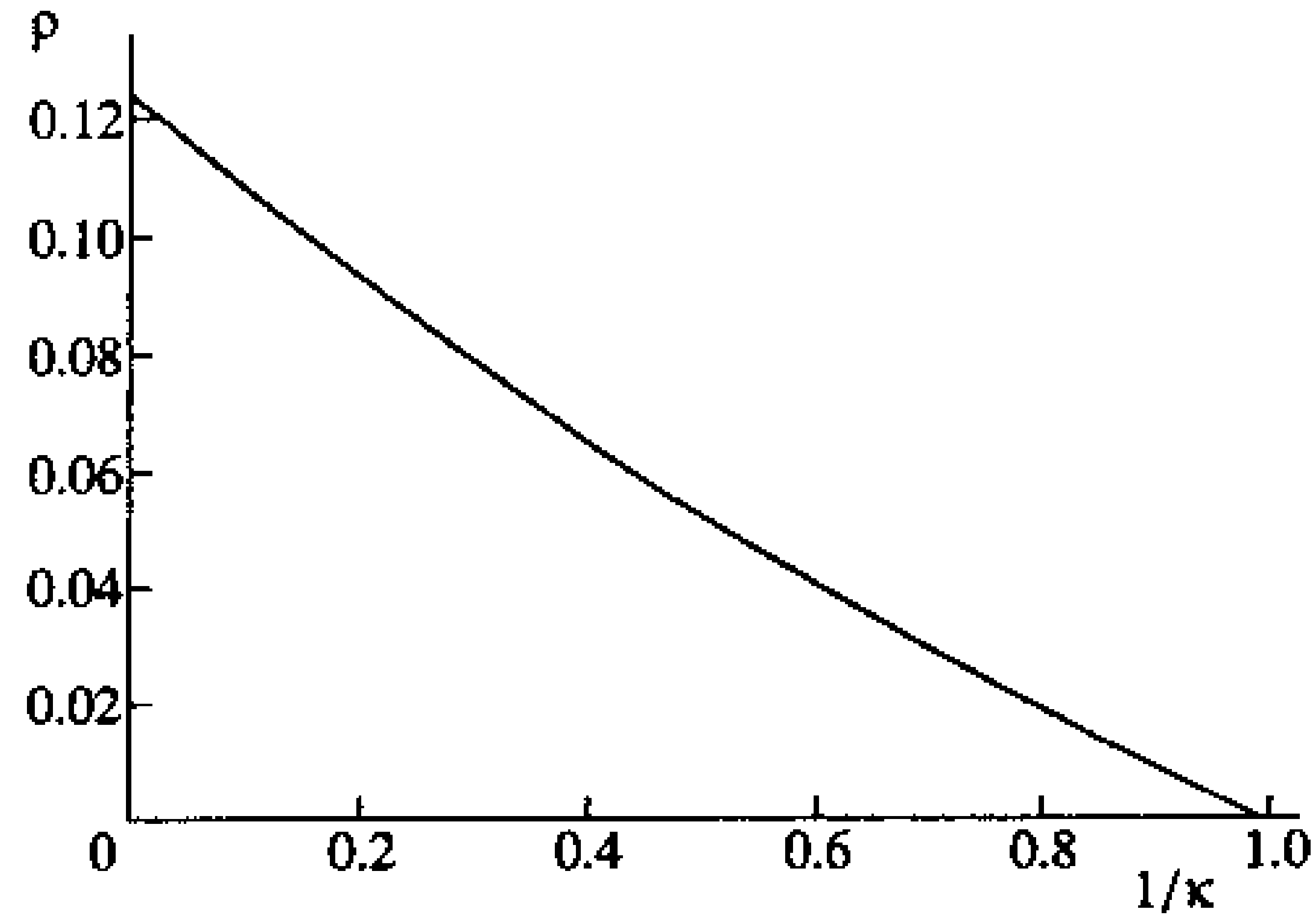

Fig. 7. Dependence of $\rho = (\nu_* - \nu_{cr})/\nu_*$ on $1/\kappa$.

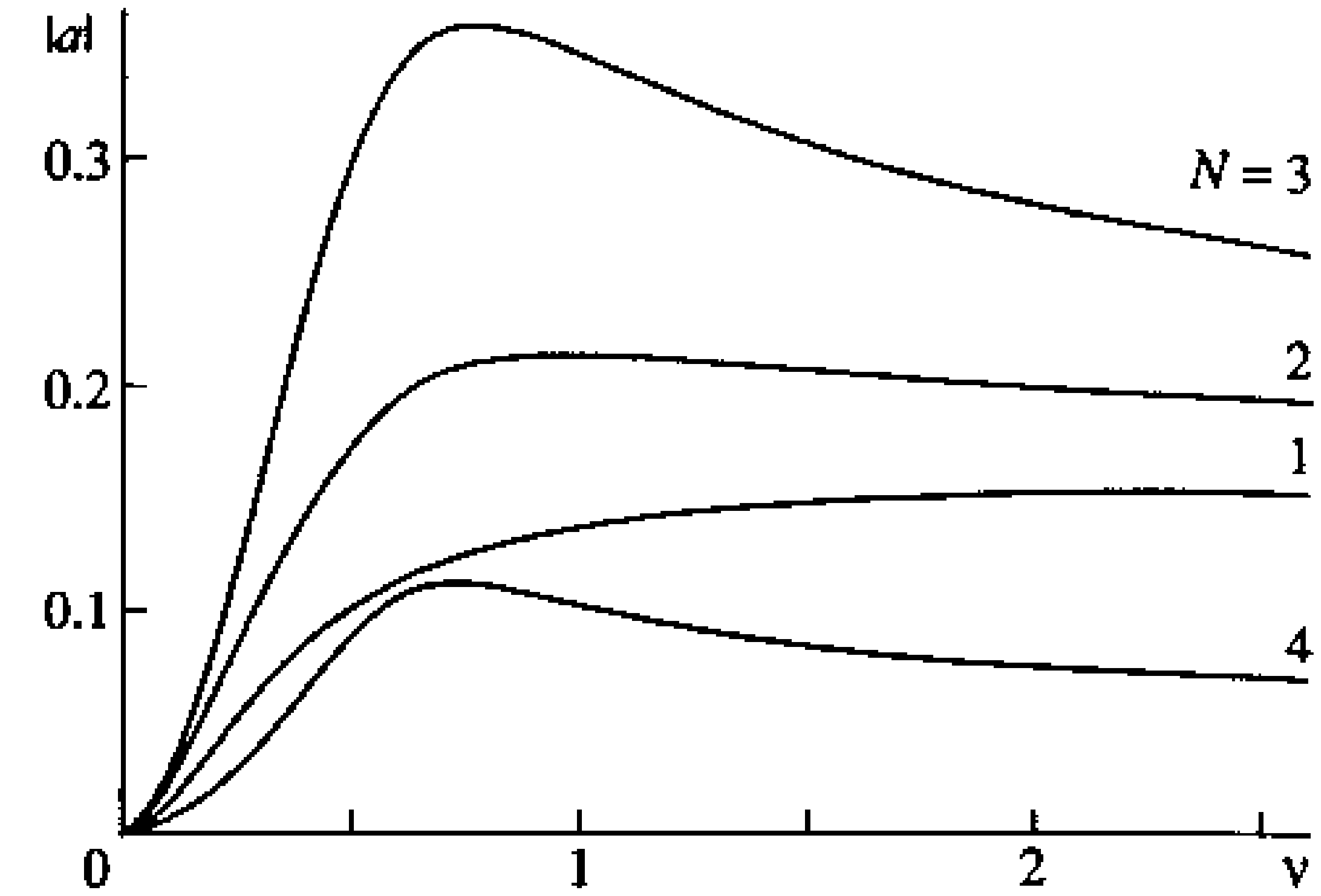

Fig. 8. Dependence of $|a|$ on $\nu = n^2\mu$ for potentials (30). The exponents N are indicated on the curves. For $N = 4$, the values of $|a(\nu)|$ are multiplied by 1/5.

$\alpha > 0$. For example, in the case of an anharmonic oscillator

$$H = \frac{1}{2}(p^2 + x^2) + gx^N, \quad -\infty < x < \infty, \tag{35}$$

the parameters of the asymptotics are known [12, 22]:

$$\alpha = \frac{N-2}{2}, \quad a = -\Gamma\left(\frac{2N}{N-2}\right)/2\Gamma^2\left(\frac{N}{N-2}\right). \tag{36}$$

However, in all of the examples considered earlier, the divergence of the $1/N$ expansion is of a factorial type: $\varepsilon^{(k)} \sim k!$ for $k \longrightarrow \infty$ (i.e., $\alpha = 1$). In order to find the reason behind such a difference in the behavior of higher orders of perturbation theory and the $1/n$ expansion, we consider the potential

$$V(r) = -\delta^{-1} r^{-\delta} - g r^{\nu}, \tag{37}$$

where $0 < \delta < 2$, $\nu > 0$, and $g \longrightarrow +0$, and which involves attraction at small distances and has a barrier. If $g = 0$, the energy levels condense to the continuum boundary $E = 0$. In the semiclassical approximation, we have

$$E_{nl} \approx -A_{nl} n^{-2\delta/(2-\delta)}, \quad n \gg 1, \tag{38}$$

and the mean radius of the bound state is given by

$$\langle r \rangle \sim (-E_{nl})^{-1/\delta} \sim n^{\frac{2}{2-\delta}} \longrightarrow \infty$$

(the constant $A_{nl} > 0$ and depends only on the ratio l/n; for example, $A_{nl} = 1/2$ at $\delta = 1$, i.e., for the Coulomb potential).

It can easily be seen that the second term in equation (37), which perturbs the energy spectrum, becomes comparable to the first term at characteristic distances $r \sim \langle r \rangle$ if

$$f \equiv n^{\frac{2(\delta+\nu)}{2-\delta}} g \sim 1. \tag{39}$$

Therefore, it is f (and not g) that is the effective coupling constant for highly excited states. In view of this statement, we can write the energy of the nlth level in the form

$$E_{nl}(g) = -A_{nl} n^{-\frac{2\delta}{2-\delta}} \varepsilon(f), \tag{40}$$

where $\varepsilon(0) = 1$. Calculating now the penetrability of the barrier for small values of g, we find with exponential accuracy that

$$\Gamma_{nl} \sim \exp\{-c_\nu g^{-\frac{1}{\nu}} - E_{nl}^{\frac{\nu+2}{2\nu}}\} = \exp\{-n/a(f)\}, \tag{41}$$

where

$$c_\nu = 2^{3/2}\pi^{1/2}\Gamma(\nu^{-1})/(\nu+2)\,\Gamma\left(\frac{\nu+2}{2\nu}\right), \tag{42}$$

$$a(f) = C f^{1/\nu}[\varepsilon(f)]^{-\frac{\nu+2}{\nu}}, \tag{43}$$

and C is a constant depending on ν, δ, and l/n, and not on the principal quantum number n. According to dispersion relations in $1/n$ (see [6]), the simple exponential dependence of the width Γ_{nl} on n [for a fixed f, see relation (41)] corresponds to the formula $\varepsilon^{(k)} \sim k!$ for the coefficients of the $1/n$ expansion for $k \longrightarrow \infty$. It should be emphasized that this conclusion does not depend on the value of δ, i.e., on the form of the potential binding a particle at small distances.

However, it follows from relation (41) for series (34) of perturbation theory that $\alpha = \nu$. In this case, $\alpha = 1$ only for $\nu = 1$, i.e., for the "Stark effect".

ACKNOWLEDGMENTS

This research was carried out with financial support from the Russian Foundation of Basic Research (grant no. 93-02-14368).

Table 2. Limiting values of the parameter $a(\nu)$ for funnel potentials

N	a_0	$\|a(\infty)\|$
1	1.5000	0.1578
2	0.6366	0.1592
3	0.5192	0.1914
4	0.4811	0.2571
5	0.4648	0.3356
6	0.4571	0.4235
8	0.4517	0.6228
10	0.4511	0.8489
15	0.4547	1.5125

Note: Here, a_0 is the coefficient in expansion (A.14) for small values of ν.

Table 3. Convergence of the values of $1/a_c$ for $\nu < \nu_{cr}$ (Yukawa potential)

k	$\nu = 0.40$	$\nu = 0.5257$
7	$-3.672 \pm i\,18.77$	$3.162 \pm i\,14.37$
10	$-1.945 \pm i\,13.51$	$0.153 \pm i\,8.135$
13	$-2.138 \pm i\,13.43$	$-0.127 \pm i\,7.996$
16	$-2.149 \pm i\,13.72$	$-3.0(-3) \pm i\,8.198$
19	$-2.310 \pm i\,14.03$	$-0.107 \pm i\,8.303$
22	$-2.310 \pm i\,13.99(*)$	$-3.0(-2) \pm i\,8.386$
25	$-2.419 \pm i\,13.91$	$1(-2) \pm i\,8.390(*)$
28	$-2.360 \pm i\,13.83(*)$	$1(-3) \pm i\,8.381$

Note: The cases when the values presented in the table correspond to the second (is closeness to zero) singularity of Borel's transformant are marked by asterisk.

APPENDIX 1

We will briefly discuss here the methods of numerical calculation of the parameters of asymptotic expansions (2) and (3) and carry out their comparative analysis.

(1) In the region $\nu > \nu_{cr}$, the coefficients of the $1/n$ expansion attain the asymptotics rapidly, and the quantities $b_k(\beta)$ depend linearly on k for $k \gg 1$:

$$b_k(\beta) \equiv \ln|\varepsilon^{(k)}/k!k^{\beta}| = k\ln a + \ln c_0 + O(1/k) \quad \text{(A.1)}$$

(see Fig. 1 for $\beta = 0$). The parameters a and c_0 can be calculated using one of the following methods.

(a) Method of linear approximation. Taking relation (A.1) into account, we can calculate a and c_0 by means of the method of least squares (however, the value of the exponent β must be specified in this case). The value of β was varied between -2 and -1. The root-mean-square deviation of the values of $b_k(\beta)$ from the right-hand side of (A.1) has the global minimum for $\beta = -1.50 \pm 0.01$, which is in agreement with the theoretical value [6] $\beta = -3/2$.

(b) Setting

$$x_k = \varepsilon^{(k)}/k\varepsilon^{(k-1)}, \quad y_k = k^2(x_{k-1}/x_k - 1), \quad \text{(A.2)}$$

we can easily see that extrapolation to the point $k^{-1} = 0$ gives $x_k \longrightarrow a$ and $y_k \longrightarrow \beta$ at a rate on the order of k^{-1}, which allows us to calculate the parameters a and β numerically. In view of the slow convergence of this method, it is suitable for obtaining preliminary estimates of a and β, whose values are then determined more precisely by using method (a).

(2) $\nu < \nu_{cr}$. The parameter a is complex-valued in this case, and hence, we must use more powerful numerical methods based on the search for complex singularities of Borel's transformant:

$$B_\beta(z) = \sum_k \frac{\varepsilon^{(k)}}{k^{\beta}k!}z^k. \quad \text{(A.3)}$$

It is well known that the radius of convergence of a power series is determined by the distance to the nearest singular point. According to relations (3) and (9), $z = 1/a_c$ and $z = 1/a_c^*$ are the poles determining the convergence of Borel's transformant. Generally speaking, they must be the singularities closest to zero. However, false poles, which may lie closer to zero than $1/a_c$ in some cases, also emerge because we are dealing with a restricted set (with $k \le 40$) of coefficients $\varepsilon^{(k)}$ (corresponding cases are marked in Table 3 by asterisks). It is important that, in varying the number of the coefficients $\varepsilon^{(k)}$ introduced in calculations, we do not observe a regular convergence of random poles, while a unique sequence of poles that always displays convergence exists. Further details of computational nature can be found in [23].

The poles of Borel's transformant can be found using the Padé–Hermite approximant (PHA). For example, diagonal PHA are defined as follows (see, for example, [9]):

$$P_L(z) + Q_L(z)\tilde{B}_\beta(z) + R_L(z)[\tilde{B}_\beta(z)]^2 = O(z^{3L+2}), \quad z \longrightarrow 0. \quad \text{(A.4)}$$

The coefficients of the polynomials P_L, Q_L, and R_L (of degree L) are uniquely determined from the first $3L+1$ coefficients of function (A.3), which are assumed to be known. Setting the right-hand side equal to zero, we obtain

$$\tilde{B}_\beta(z) = \frac{-Q_L \pm (Q_L^2 - 4P_LR_L)^{1/2}}{2P_L}. \quad \text{(A.5)}$$

This expression has the following singularities: the roots of the denominator (poles) and the radicand in the numerator (branch points), among which we must choose the one closest to the frame origin (the asymptotic parameter c_0 is determined as a residue at the pole $z = 1/a_c$). The disadvantage of this method lies in the fact that the parameter β_c must be preset. Numerical analysis shows that the dependence on β_c is rather weak, and hence, we can set $\beta_c = 0$ at the beginning of calculations and then vary its value within a certain interval. According to our estimates, $\beta_c \approx -1.5 \pm 0.5$.

We will illustrate the process of convergence for the Yukawa potential. Because we analyze only diagonal PHAs, the number of coefficients of the $1/n$ expansion introduced in calculations is $k = 3L + 1$ [see Table 3, which contains the values of $1/a_c$ calculated by equations (A.4) and (A.5)]. The relative error in the calculation of $1/a_c$ amounts to 1 to 2%. This method was used for calculating the curves in Figs. 2 and 3.

In the case of Hulthén potential (12), the situation is the same, and only the numerical values of the parameters ν_{cr} and ν_* are different.

APPENDIX 2

Calculation of $a(\nu)$ for Funnel Potentials

It should be noted above all that the funnel potential has only a discrete spectrum for $0 < g < \infty$. Therefore, it is convenient to change the sign of the coupling constant and to assume first that $g < 0$ (this trick was proposed for the first time by Dyson [11]). Then, potential (30) has a barrier whose penetrability determines the width of quasi-stationary states and the asymptotic parameter $a(\nu)$:

$$a^{-1} = \int_C \left[\frac{1}{x^2} - \frac{2}{\nu x}f(x) - \frac{\varepsilon^{(0)}}{\nu^2}\right]^{1/2} dx, \qquad \text{(A.6)}$$

where the contour C embraces the turning points (including the point x_0), and $f(x)$ is the screening function from expression (5). In the case under investigation, $f(x) = 1 + N^{-1}x^{N+1}$. Here, the point of equilibrium $x_0(\nu)$ can be determined from the equation $x - x^{N+2} = \nu$ and is real-valued for $\nu < \nu_*$, where

$$\nu_* = (N+1)(N+2)^{-\frac{N+2}{N+1}}, \quad \nu = n^2(-g)^{\frac{1}{N+1}}. \text{(A.7)}$$

We will consider in detail two limiting cases $\nu \longrightarrow 0$ and $\nu \longrightarrow \infty$.

(1) For $\nu \longrightarrow 0$, we have

$$x_0 = \nu + \nu^{N+2} + O(\nu^{2N+3}),$$
$$\varepsilon^{(0)} = -1 - 2N^{-1}\nu^{N+1} + \ldots, \qquad \text{(A.8)}$$

where

$$(2a)^{-1} = \left(\frac{x_0}{\nu}\right)^{1/2} \times \int_1^{t_2} \{ (1-t^{-1})^2 - x_0^{N+1}p_N(t)t^{-2} \}^{1/2} dt. \qquad \text{(A.9)}$$

Here, $t = 1$ and t_2 are turning points,

$$t = x/x_0, \quad t_2 = (N/2)^{1/N}\nu^{-\frac{N+1}{N}},$$
$$p_N(t) = 2N^{-1}t^{N+2} - (N+2)N^{-1}t^2 + 1 = (1-t)^2 q_N(t), \qquad \text{(A.10)}$$

where[8)]

$$q_1 = 2t + 1, \quad q_2 = (t+1)^2,$$
$$q_3 = \frac{2}{3}(t^3 + 2t^2 + 3t + 3/2),$$
$$q_4 = \frac{1}{2}(t+1)^2(t^2+2),$$
$$q_6 = \frac{1}{3}(t+1)^2(t^4 + 2t^2 + 3), \ldots.$$

Because $t_2 \longrightarrow \infty$ for $\nu \longrightarrow 0$, we can write the integral in (A.9) in the form

$$J = \int_1^{\bar{t}} (1 - t^{-1})\,dt + \int_{\bar{t}}^{t_2} \{ (1-ct^N)^{1/2} - t^{-1}(1-ct^N)^{-1/2} + \ldots \} dt, \qquad \text{(A.11)}$$

where $c = 2N^{-1}\nu^{N+1}$, $\bar{t}$ is the sewing point, $1 \ll \bar{t} \ll t_2$, and we disregard the terms vanishing for $\nu = 0$. Taking into account the values of the integrals

$$C_N = \int_0^1 (1-t^N)^{1/2} dt = \pi^{1/2}\Gamma(N^{-1})/(N+2)\Gamma(\frac{N+2}{2N}),$$
$$\int_z^1 \frac{dt}{t}(1-t^N)^{1/2} = \frac{2}{N}\operatorname{arctanh}(1-z^N)^{1/2} = -\ln z + \frac{2\ln 2}{N} + O(z^N), \quad z \longrightarrow 0, \qquad \text{(A.12)}$$

we obtain

$$J = \left(\frac{N}{2}\right)^{1/N} C_N \nu^{-\frac{N+1}{N}} + \frac{N+1}{N}\ln\nu + O(1) \qquad \text{(A.13)}$$

(the arbitrary sewing point $\bar{t}$ does not appear in the final result as expected). Finally, we obtain

$$a(\nu) = a_0\nu^{\frac{N+1}{N}}\left[1 - a_1\nu^{\frac{N+1}{N}}\ln\nu + O(\nu^{\frac{N+1}{N}})\right], \quad \nu \longrightarrow 0, \qquad \text{(A.14)}$$

where

$$a_0 = \frac{N+2}{2^{\frac{N-1}{N}}\pi^{1/2}N^{\frac{N+1}{N}}}\Gamma\left(\frac{N+2}{2N}\right)/\Gamma\left(\frac{N+1}{N}\right), \qquad \text{(A.15)}$$

$$a_1 = 2(N+1)N^{-1}a_0. \qquad \text{(A.16)}$$

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8) If N is an integer, $q_N(t)$ is a polynomial of the Nth degree. It should be noted that formulas (A.9) and (A.14) remain valid for an arbitrary $N > 0$.

It should be noted that $\lambda = \nu^{N+1} = -n^{2N+2}g$ is an effective coupling constant for highly excited states (for example, for $N = 1$, the constant λ is similar to the "reduced" electric field $F = n^4\varepsilon$ in the Stark effect theory [9]). Going over to the case $g > 0$, we must make the substitution $\lambda \longrightarrow \lambda\exp(-i\pi)$. In this case, formula (A.14) remains valid, but the parameter $a(\nu)$ becomes complex-valued.

(2) For $\nu \longrightarrow \infty$, we have

$$x_0(\nu) \approx (\lambda e^{-i\pi})^{\frac{1}{(N+1)(N+2)}} \sim \nu^{\frac{1}{N+2}}, \qquad (A.17)$$

the second term becomes dominant in (A.9), and

$$\frac{1}{2a(\infty)} = J_N \equiv \int_{\tau}^{1} [p_N(t)]^{1/2}\frac{dt}{t}, \qquad (A.18)$$

where τ is the root of the polynomial $p_N(t)$ that differs from the value $t_0 = 1$ for which the absolute value of the integral is minimum. The contour of integration circumvents the pole $t = 0$ from above.

It should be noted that, for $N = 1$, 2, 4, and 6, the integral (A.18) can be evaluated analytically. For $N = 1$, we have $\tau = -1/2$, and

$$J_1 = \oint_{-1/2}^{1} (2t+1)^{1/2}(t^{-1}-1)\,dt - i\pi = 3^{1/2} - \ln(2+3^{1/2}) - i\pi, \qquad (A.19)$$

while for $N = 2$, we have

$$\tau = -1, \quad J_2 = \oint_{-1}^{1}(1-t^2)\frac{dt}{t} - i\pi = -i\pi. \qquad (A.20)$$

The polynomial $q_4(t)$ has the roots $t_1 = t_2 = -1$ and $t_{3,4} = \pm i\sqrt{2}$. Substituting these roots into (A.18), we obtain the following possible values:

$$\frac{1}{a(\infty)} = -2\pi i, \quad 2\ln(3^{1/2} - 2^{1/2}) \mp i\pi, \qquad (A.21)$$

from which the second pair (i.e., $\tau = t_3$) corresponds to the maximum value of $|a(\infty)|$.

Finally, we consider the case when $N = 6$:

$$J_6 = 3^{-1/2}\int_{\tau}^{1}(t^8 - 4t^2 + 3)^{1/2}\frac{dt}{t}. \qquad (A.22)$$

The possible values of τ are $t_{1,2} = -1$ [the double root of the polynomial $q_6(t)$] and $t_j^2 = -1 \pm i\sqrt{2}$. In the former case, the integral in the sense of the principal value in (A.18) vanishes, and $J_6 = -i\pi \cdot 3^{-1/2}$. For the remaining roots, it is convenient to go over to the variable $u = t^2$, and the integral is reduced to the trivial integral

$$J_6' = \frac{1}{2\sqrt{3}}\int_{\tau^2}^{1}(u^2+2u+3)^{1/2}(u^{-1}-1)\,du = \ln(1+\sqrt{2}) \mp i\pi/4, \qquad (A.23)$$

satisfying the inequality at $|J_6| > |J_6'|$. Therefore,

$$a(\infty) = \frac{1}{2J_6'} = [2\ln(1+\sqrt{2}) - i\pi/2]^{-1}. \qquad (A.24)$$

For any N, the integral in (A.18) can be evaluated numerically; the results of the calculations are presented in Table 2. Note that, for $N = 1$ and 2, we can obtain closed analytic expressions for $a(\nu)$ see formulas (B.8) and (B.10) from [6]}. Expansion (A.14) and formulas (A.19) and (A.20) are in agreement with these formulas in two limiting cases.

The above formulas are valid for $g < 0$. The results for funnel potential (30) can be obtained from these formulas by analytic continuation in ν. For $\nu = \infty$, the quantities $a(\infty)$ and $a^*(\infty)$ make a contribution to the asymptotics of the coefficients appearing in the $1/n$ expansion, leading eventually to formula (3).

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