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NUCLEI
Theory

Coulomb Corrections to Nuclear Scattering Lengths and Effective Ranges for Weakly Bound Systems

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Abstract – A procedure is considered for extracting the purely nuclear scattering length a_s and effective range r_s (which correspond to a strong-interaction potential V_s with disregarded Coulomb interaction) from the experimentally determined nuclear quantities a_{cs} and r_{cs}, which are modified by Coulomb interaction. The Coulomb renormalization of a_s and r_s is especially strong if the system under study involves a level with energy close to zero (on the nuclear scale). This applies to formulas that determine the Coulomb renormalization of the low-energy parameters of s scattering ($l = 0$). Detailed numerical calculations are performed for coefficients appearing in the equations that determine Coulomb corrections for various models of the potential $V_s(r)$. This makes it possible to draw qualitative conclusions about the dependence of Coulomb corrections on the form of the strong-interaction potential and, in particular, on its small-distance behavior. A considerable enhancement of Coulomb corrections to the effective range r_s is found for potentials with a barrier.

1. INTRODUCTION

Calculation of Coulomb corrections to the parameters of low-energy scattering, such as the scattering length $a_s^{(l)}$, the effective range $r_s^{(l)}$, and the shape coefficient P (at $l = 0$, where l is the angular momentum), is important for various applications. The related problems were considered by many authors (see, for example, [1 - 12] and references therein). The general expressions for Coulomb corrections in a state with arbitrary angular momentum were obtained in [10, 11]. The corrections to the scattering length for s-wave scattering ($l = 0$) were considered previously by Schwinger [1] (see also [3, 4]). Specific numerical calculations were performed for a limited number of potentials [11], as this work requires a lot of machine time. This hampers detailed investigation of the dependence of Coulomb corrections on the form of the strong-interaction potential $V_s(r)$ and, in particular, on its small-distance behavior.

In this study, we aim at filling this gap. The exposition is as follows. In Section 2 and Appendix A, we present a compendium of formulas that are used to calculate Coulomb corrections to the s-wave scattering length and to the effective range ($l = 0$) for an arbitrary local potential $V_s(r)$. In Sections 3 and 4, we give an account of numerical results obtained for a large number of short-range potentials, including the Yukawa and Hulthén potentials, which are often used in atomic and nuclear physics (the results for Coulomb corrections in the ground state are presented in Section 3, and excited states are considered Section 4). Section 5 is devoted to Coulomb corrections for potentials with a barrier. The results are briefly summarized in Section 6. Some details of the calculations are described in Appendices A and B.

In this study, we confine our analysis to $l = 0$ states and to the resonant case in which there is a shallow nuclear level (real, virtual, or quasistationary) in the strong-interaction potential V_s; here, Coulomb corrections to the s-scattering length are most significant. In the following, we omit the index l and set $a_s^{(0)} \equiv a_s$, $a_{cs}^{(0)} \equiv a_{cs}$, etc.

2. COULOMB RENORMALIZATION OF THE PARAMETERS OF LOW-ENERGY s SCATTERING

The leading (logarithmic) term in the Coulomb correction to the scattering length is determined by the Schwinger formula (see [1] and equations (342) - (348) in [3]). In the resonant case, this formula assumes the form

$$\frac{1}{a_{cs}} - \frac{1}{a_s} = \frac{2\sigma}{a_B} J_0, \qquad (1)$$

where

$$J_0 = \int_0^\infty [\chi_0^2(r) - \theta(r-R)]\frac{dr}{r} - \ln\frac{2R}{a_B} - 2C, \qquad (1a)$$

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a_s and a_{cs} are the nuclear scattering length (for disregarded Coulomb interaction – that is, for $a_B \longrightarrow \infty$) and nuclear scattering length modified by Coulomb interaction, $a_B = \hbar^2/me^2|Z_1Z_2|$ is the Bohr radius, $\sigma = \text{sgn}(Z_1Z_2) = \pm 1$, r_N is the strong-interaction range, $C = 0.5772\ldots$ is the Euler constant, $\theta(x) = (1/2)[1 + \text{sgn}x]$ is the Heaviside step function, and $\chi_0(r)$ is the wave function in the strong-interaction potential V_s for the coupling-constant value at which an s level arises in this potential.[3] Hereafter, we use the system of units in which $\hbar = m = 1$, where m is the reduced mass of the system under study.

We set

$$V_s(r) = -\frac{g}{2r_N^2}v(r/r_N), \tag{2}$$

where g is the dimensionless coupling constant, and the function $v(x)$ determines the form of the strong-interaction potential. The wave function χ_0 and the coupling constant g_s that correspond to the emergence of an s level are determined from the Schrödinger equation

$$\chi_0'' + g_s v(x)\chi_0 = 0, \quad x = r/r_N, \tag{3}$$

with boundary conditions

$$\chi_0(x) \propto x \quad \text{for} \quad x \longrightarrow 0, \qquad \lim_{x\to\infty}\chi_0(x) = 1 \tag{3a}$$

(the last condition fixes the normalization of χ_0).

Equation (1a) involves an arbitrary parameter $R > 0$ that is necessary for convergence of the integral. By differentiating (1a) with respect to R, we can easily show that J_0 is independent of R.

Corrections to the Schwinger formula in the approximation linear in

$$\delta = r_s/a_B, \quad \delta_s = r_s/a_s, \tag{4}$$

were obtained in [10, 11]. It is convenient to represent the result in the form[4]

$$\frac{1}{a_{cs}} - \frac{1}{a_s} = \frac{\sigma}{a_B}\{\ln\delta^2 + 2[c_0 - \sigma c_1\delta - b_1\delta_s + c_2\delta^2 + b_2\delta\delta_s + \ldots]\}, \tag{5}$$

where $c_0 = 2C + \ln(2r_C/r_s)$, while r_s and r_C are the effective range and Coulomb radius of the system that correspond to the emergence of the s level; that is, they are given by

$$r_s = 2r_N\int_0^\infty [1 - \chi_0^2(x)]\,dx,$$
$$r_C = r_N \exp\left\{\ln R + \int_0^\infty [\theta(x-R) - \chi_0^2(x)]\,\frac{dx}{x}\right\}. \tag{6}$$

Thus, r_s and r_C are completely determined by the wave function χ_0. Explicit expressions for the dimensionless coefficients b_1 and c_1, which additionally depend on the form of the potential $v(x)$, are presented in Appendix A, where we also describe some details of numerical calculations.

The coefficients b_2 and c_2 can be found for some extremely simple models (for example, for a δ-function potential); however, general expressions for these coefficients in an arbitrary potential $V_s(r)$ are not known at present. In expansion (5), we will henceforth discard terms of order δ^2 and $\delta\delta_s$, as well as higher-order terms. The smaller the parameters δ and δ_s, the more accurate results are obtained in this approximation. The condition $\delta \ll 1$ is fulfilled well for extremely light hadronic systems (for example, in a proton–antiproton atom and in the pp system, we have $r_s < 3$ fm, $a_B = 57.6$ fm, and $\delta < 0.05$). On the other hand, the condition $|\delta_s| \ll 1$ implies that there is a shallow level (on the nuclear scale) in the system under study because in this case, the scattering length considerably exceeds the effective range.[5] However, this is not always the case.

In contrast to (5), the Coulomb renormalization of the effective range does not involve (for $l = 0$) the large logarithm $\ln\delta$ [11, 19, 20]; that is,

$$r_{cs}/r_s = 1 - \sigma c_1'\delta - b_1'\delta_s + O(\delta^2, \delta\delta_s), \tag{7}$$

where r_{cs} is the Coulomb-modified nuclear effective range corresponding to the emergence of the s level in the system ($g = g_{cs}$), and r_s the effective range only for $V_s(r)$ (at $g = g_s$). The difference of the coupling constants g_s and g_{cs} is given by (A.1), and the dimensionless coefficients b_1' and c_1' are determined by equations (A.9) and (A.10).

[3] For $r \gg r_N$, the total potential has the form $V(r) = V_s + V_C = \sigma/a_B r$, where $\sigma = 1$ for Coulomb repulsion (dt, d^3He, $\alpha\alpha$, etc., systems) and $\sigma = -1$ for attraction (for example, for hadronic atoms $\bar{p}p$, K^-p, etc.).

[4] This expression contains only terms of the form δ^k and $\delta^{k-1}\delta_s$, $k \geq 1$.

[5] It is the situation that followed from the analysis [10, 13] of first measurements [14] of the nuclear shift of the $\bar{p}p$-atom ground state (however, these experimental data have not been confirmed by later studies [15, 16]). The current value of the nuclear shift ΔE_{1s} is about 1/15 of the spacing between the 1s and 2s levels of the $p\bar{p}$ atom, while the scattering length averaged over spin states is $a_s = -0.88 + i0.84$ fm (for details, see [17]). Under such conditions, there is obviously no shallow nuclear level in the $p\bar{p}$ system. On the other hand, we have $\delta_s \approx -0.11$ for the pp system. The smallness of δ_s is explained by the existence of a shallow virtual $^1S_0$ level in the np system and, by virtue of isotopic invariance, in the nn system as well (see, for example, [18]).

Table 1. Parameters for the attraction potentials (2)

No.	$v(x)$	g_s	x_s	ρ	c_0	c_1	b_1	c'_1	b'_1	β
1	$e^{-x}x^{-1}$	1.6798	2.1200	0.3639	0.8367	0.842	1.056	−0.177	0.872	0.441
2	$(e^{2x}-e^{x})^{-1}$	2.5913	1.3169	0.3671	0.8455	0.889	1.038	−0.064	0.839	0.454
3	$(2e^{-x}-e^{-2x})x^{-1}$	1.0809	2.3610	0.37427	0.8648	0.974	1.000	0.079	0.792	0.587
4	$(e^{x}-1)^{-1}$	1.0000	3.0000	0.37431	0.8649	0.983	1.000	0.134	0.778	0.500
5	$e^{-x}x^{-1/2}$	1.7251	2.8376	0.3831	0.8880	1.079	0.957	0.271	0.731	0.295
6	$1/\mathrm{sh}x$	0.6414	3.3318	0.3849	0.8929	1.108	0.947	0.355	0.704	0.670
7	$e^{-x}(1+x)^{-1}$	3.6648	2.6954	0.3870	0.8983	1.120	0.940	0.328	0.711	0.144
8	e^{-x}	1.4458	3.5408	0.3951	0.9190	1.215	0.903	0.512	0.651	0.270
9	$e^{-x^2}x^{-1}$	1.7510	1.0698	0.3961	0.9214	1.243	0.895	0.630	0.613	0.732
10	$1/\cosh x$	1.7716	3.6516	0.3992	0.9294	1.257	0.887	0.578	0.629	0.483
11	$2e^{-x}-e^{-2x}$	0.8096	3.7054	0.3996	0.9302	1.262	0.885	0.590	0.625	0.453
12	$(e^{x}+1)^{-1}$	1.7206	3.7831	0.4007	0.9332	1.275	0.880	0.614	0.616	0.208
13	$e^{-x}(1+x)$	0.4673	4.5494	0.4036	0.9404	1.307	0.869	0.675	0.596	0.630
14	$(\cosh x)^{-2}$	2.0000	2.000	0.4655	0.9449	1.323	0.862	0.693	0.589	0.333
15	$x/\mathrm{sh}x$	0.3285	4.8561	0.4074	0.9496	1.345	0.855	0.738	0.573	0.830
16	$\exp(-x^2)$	2.6840	1.4352	0.4184	0.9762	1.457	0.814	0.936	0.502	0.333
17	$\exp(-x^4)$	2.8924	1.0687	0.4302	1.0041	1.564	0.778	1.095	0.442	0.403

Note: Here and in Table 2, the potential is taken in the form (2), $x = r/r_N$, $x_s = r_s/r_N$, and $\rho = r_C/r_s$. All quantities refer to the ground-state level. The parameter β is defined in (A.1) and (A.5).

Table 2. Finite potentials

No.	$u(x)$	g_s	x_s	ρ	c_0	c_1	b_1	c'_1	b'_1	β
1	$(1-x)^2/x$	4.7638	0.4855	0.3947	0.9178	1.237	0.900	0.658	0.606	0.597
2	$(1-x)/x$	3.0942	0.6260	0.4043	0.9420	1.338	0.858	0.819	0.546	0.687
3	$1/x$	1.4458	0.8722	0.4218	0.9842	1.500	0.793	1.043	0.457	1.000
4	$1-x$	7.8373	0.7870	0.4286	1.0000	1.553	0.780	1.089	0.443	0.202
5	$1-x^2$	5.1217	0.8240	0.4319	1.0081	1.580	0.772	1.122	0.431	0.294
6	$1-x^4$	3.7736	0.8694	0.4350	1.0152	1.605	0.765	1.153	0.419	0.377
7	1	2.4674	1.0000	0.4386	1.0234	1.634	0.757	1.189	0.405	0.500
8	δ-function potential	1.0000	1.3333	0.4549	1.0599	1.750	0.750	1.312	0.375	1.000
9	Breit potential	1.0000	2.0000	0.5000	1.1544	2.000	1.000	1.500	0.500	1.000

Let us proceed to the discussion of numerical results.

3. NUMERICAL RESULTS

Using the above formulas, we calculated numerically the Coulomb corrections to a_s and r_s for potentials of the form (2) (see Table 1, in which all the quantities are presented for the ground-state level $n = 1$). The effective range r_s and the Coulomb radius r_C are in direct proportion to the parameter r_N, while the coefficients c_0, c_1, b_1, ... are independent of this parameter; these coefficients are determined by the form of the function $v(x)$ (that is, by the form of the strong-interaction potential) and, of course, by the level number n. In Table 1, the potentials are arranged in order of increasing $r_C/r_s \equiv \rho$. This presentation of data enables us to reveal a remarkable correlation between this ratio and the coefficients c_1 and b_1. The corresponding quantities for finite potentials (that is, $V_s(r) \equiv 0$ for $r > r_N$) are presented in Table 2, where we introduced the notation $v(x) = u(x)\theta(1-x)$. Analysis of our numerical results leads to the following conclusions.

(1) The coupling constant g_s and the effective range $r_s = x_s r_N$ undergo large variations as we go over from one potential to another, but they do not exhibit any regularity. At the same time, the coefficients c_0, c_1, etc., which control the Coulomb corrections in (5) and (7),

are much more stable (for example, the coefficient c_0 varies between ≈0.85 and 1.05 for all potentials from Table 1, which includes potentials that are finite at the origin and potentials that have a Coulomb singularity at this point).

(2) The ratio $\rho = r_C/r_s$ is the main parameter that determines Coulomb corrections (especially the coefficients c_0, c_1, and b_1). Given the parameter ρ, we can predict the values of these coefficients to a high accuracy. On the other hand, the scattering lengths are sharply varying functions of the coupling constant g and of the form of potential.

(3) From comparison of data presented in Tables 1 and 2, it can be seen that the results obtained with finite potentials are qualitatively similar to those for smooth potentials, although the ρ values are greater for the former than for the latter by some 10%. Table 2 also includes two potentials that are determined by boundary conditions. These are the δ-function potential $v(x) = \delta(x-1)$, which correspond to the condition (at $r = r_N$)

$$r\frac{d\varphi}{dr}\bigg|_{r_N+0} - r\frac{d\varphi}{dr}\bigg|_{r_N-0} = -g\varphi(r_N), \quad (8)$$

and the Breit potential [21, 22], for which we have

$$r\frac{d\varphi}{dr}\bigg|_{r_N} = -g\varphi(r_N), \quad \varphi(r) \equiv 0 \quad \text{for} \quad 0 < r < r_N. \quad (9)$$

In the last case, all quantities that determine the Coulomb renormalization of a_s and r_s attain their maximum values.

(4) Nuclear-physics calculations are often performed with the Woods–Saxon potential

$$v = \left[\exp\left(\frac{r-R}{a}\right) + 1\right]^{-1} = (e^{x-d}+1)^{-1}, \quad (10)$$

where $d = R/a$, a is the diffuseness parameter of a nucleus, and R is its radius (for heavy nuclei, we have $a = 0.53$ fm and $d \approx 15$ [23]). With increasing d, a smooth transition from the case $d = 0$ (potential no. 12 from Table 1) to a rectangular well ($d \longrightarrow \infty$) occurs. The resulting variations of all Coulomb corrections prove to be monotonic, the coefficients c_1, b_1, etc., being virtually constant for $d \gtrsim 10$.

(5) Inspection of data presented in Tables 1 and 2 reveals that, for the Yukawa potential, the parameter ρ has a maximum value and the coefficient c_1' is negative (otherwise, we have $c_1' > 0$). Unfortunately, general expressions (A.10) and (A.11) for c_1' are rather cumbersome; in particular, no definitive conclusions can be drawn about the sign of this coefficient. To clarify the situation, we considered the one-parameter family of potentials

$$v(x, \lambda) = \lambda e^{-x}(1 - e^{-\lambda x})^{-1}, \; -\infty < \lambda < \infty, \quad (11)$$

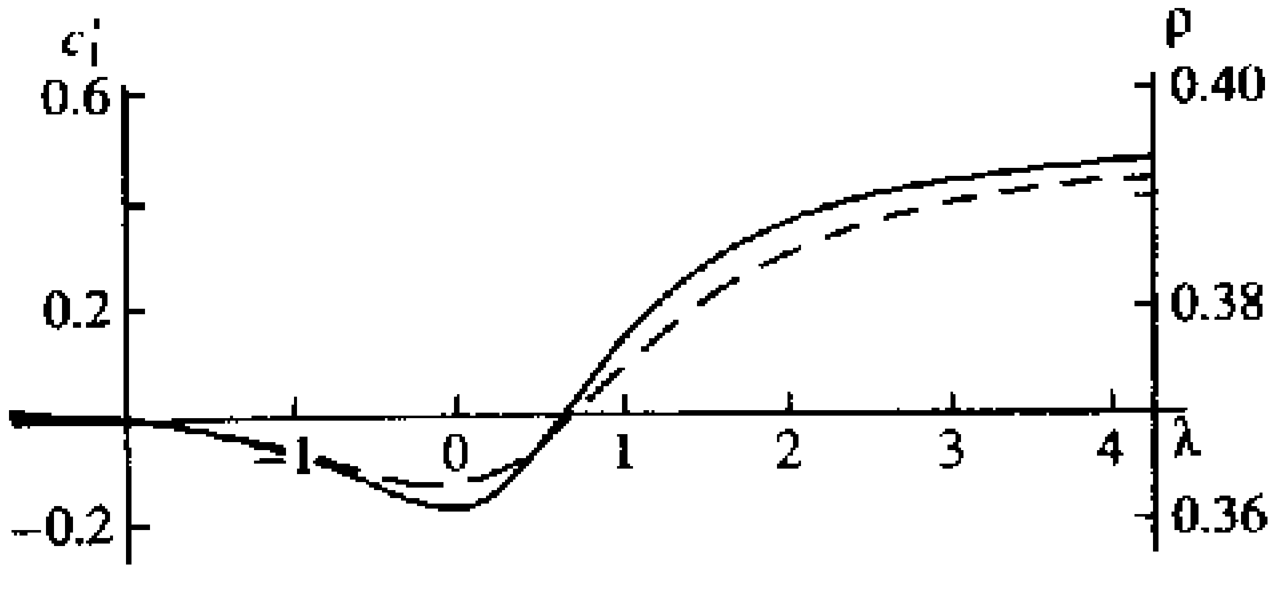

Fig. 1. Coefficient c_1' (solid curve) and ratio ρ (dashed curve) for the potentials (11) as functions of the parameter λ.

which includes the Yukawa potential (at λ = 0); the Hulthén potential (at λ = 1); and the potentials 1/sinh$x$ (λ = 2), $v(x) = 1/(\sinh x) + 1/(\cosh x)$ (λ = 4), and $v(x) = (e^{2x} - e^x)^{-1}$ (λ = −1). All these potentials have a Coulomb singularity at the origin and exponentially decrease at infinity; that is,

$$v(x, \lambda) = \begin{cases} x^{-1} + (\lambda - 2)/2 + O(x), & x \longrightarrow 0 \\ |\lambda|\exp(-\nu x), & x \longrightarrow \infty, \end{cases} \quad (11a)$$

where ν = 1 for λ > 0 and ν = |λ| + 1 for λ < 0 (if λ = 0, we have $v(x, 0) = e^{-x}/x$ for $x \longrightarrow \infty$). For $\lambda \longrightarrow +\infty$, the potential (11) is equivalent to an exponential potential.

The calculated values of ρ and c_1' are presented in Fig 1. It can be seen from this figure that $c_1' < 0$ in a relatively broad range of λ ($\lambda_0' < \lambda < \lambda_0$, where $\lambda_0 \approx 0.656$ and $\lambda_0' = \lambda_0/(\lambda_0 - 1) \approx -1.907$). Figure 1 also shows that ρ and c_1' attain minimum values at λ = 0 (that is, for the Yukawa potential).

(6) Tables 1 and 2 include potentials that are finite at the origin and those that have a Coulomb singularity at this point. Let us discuss the dependence of Coulomb corrections on the behavior of the strong-interaction potential at small distances in a more general case. Assuming a power-law behavior of the potential $V_s(r)$ at the origin, we set

$$v(x) = x^{-\alpha}\exp(-x) \quad (12)$$

or

$$v(x) = x^{-\alpha}\theta(1 - x), \quad (13)$$

where $0 \le \alpha < 2$. Figure 2 shows that the effective range of the 1s state monotonically decreases with increasing α [that is, with increasing strength of the singularity of $V_s(r)$ for $r \longrightarrow 0$]. The rate $\rho = r_C/r_s$ and the coefficient c_0 in (5) are virtually constant in the range $0 \le \alpha \le 1$. Changes in these parameters become noticeable beginning from $\alpha \gtrsim 1.5$. This is obviously associated with the fact that at α = 2, collapse into the origin occurs in the potentials under study [18]. The resulting narrowing of the region around $r = 0$ in which the wave function of

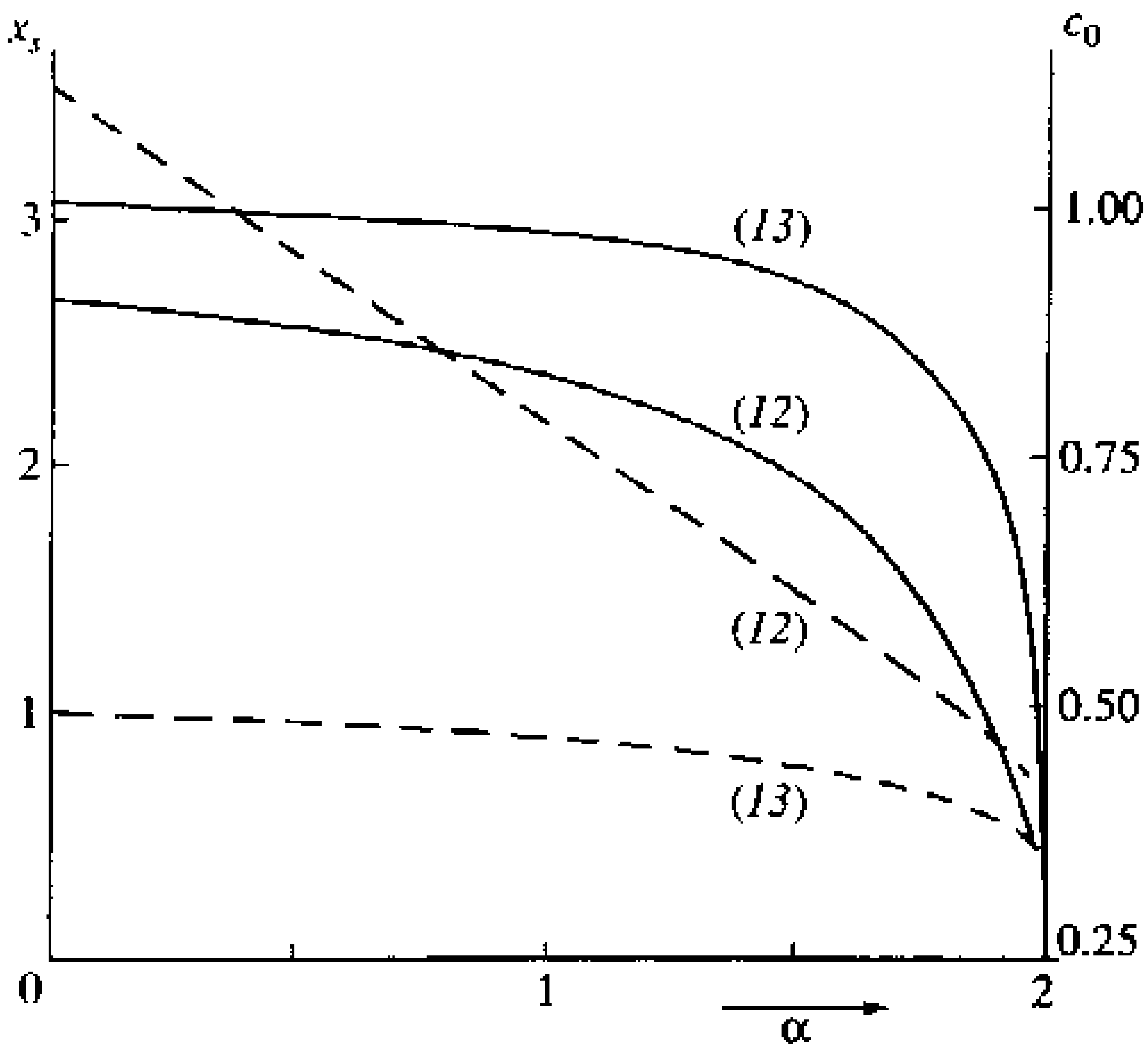

Fig. 2. Coefficient c_0 (solid curves) and effective range (dashed curves) as functions of α for potentials with a power-law singularity at the origin. Curves (*12*) and (*13*) correspond to potentials defined by equations (12) and (13) and refer to the ground state.

the bound state is localized explains a rapid decrease in r_s and r_C as $\alpha \longrightarrow 2$.

4. EXCITED STATES

From the physical point of view, it is natural to expect that a shallow nuclear level (if it exists in the system) corresponds to $n \sim 1$. For this reason, the ensuing analysis of Coulomb corrections for states with $n \gg 1$ is of purely theoretical interest. Nonetheless, it is useful for getting a more comprehensive idea of the phenomenon.

We confine our analysis to two short-range potentials (the Yukawa and Hulthén potentials). Numerical calculations for the Yukawa potential were performed up to $n = 10$. The exact solution is known for the Hulthén potential [24]. With the aid of this solution, the calculation of r_s, r_C, and Coulomb corrections can be reduced to a purely algebraic procedure [11]. This makes it possible to obtain numerical results for much larger values of n.

It can easily be seen that $g_s \propto n^2$ for ns levels with $n \gg 1$. Introducing the notation $\lambda_n = g_s/n^2$ and using the WKB method, we obtain

$$\lambda_\infty = \lim_{n\to\infty}\lambda_n = \left\{\pi/\int_0^\infty \sqrt{v(x)}dx\right\}^2. \tag{14}$$

In particular, we have $\lambda_\infty = \pi/2$ for the Yukawa potential and $\lambda_\infty = 1$ for the Hulthén potential (from the exact solution, it follows that in the latter case, λ_n is independent of n).

The results of calculations are given in Table 3, which displays r_s and r_C values and the most interesting Coulomb corrections. We can see that r_s and r_C logarithmically increase with increasing level number n. This is associated with the fact that with increasing n, the wave function $\chi_0(r)$ approaches the asymptotic

Table 3. Coulomb corrections for excited *ns* states

n	λ_n	x_s	ρ	c_0	c_1	b_1	c_1'	b_1'
Yukawa potential								
1	1.6798	2.120	0.3639	0.837	0.842	1.056	−0.177	0.872
2	1.6118	3.962	0.4001	0.932	1.429	0.962	0.935	0.715
3	1.5936	5.143	0.4195	0.979	1.584	0.954	1.140	0.639
4	1.5857	4.019	0.4313	1.007	1.661	0.955	1.226	0.611
5	1.5816	6.717	0.4392	1.025	1.708	0.957	1.275	0.594
10	1.5747	8.971	0.4584	1.068	1.811	0.968	1.368	0.559
Hulthén potential								
1	1	3	0.3743	0.865	0.983	1.000	0.134	0.778
2	1	11/2	0.3974	0.925	1.453	0.904	1.007	0.653
3	1	7	0.4161	0.971	1.590	0.910	1.175	0.608
4	1	97/12	0.4281	0.999	1.662	0.920	1.249	0.586
5	1	134/15	0.4365	1.019	1.707	0.928	1.292	0.573
10	1	3659/315	0.4572	1.065	1.811	0.952	1.377	0.546
15	1	1.321(1)	0.4661	1.084	1.852	0.962	1.407	0.537
20	1	1.434(1)	0.4712	1.095	1.876	0.971	1.423	0.535

Note: Here and in Tables 4 and 5, the exponents of the values are presented in parentheses (for example, 1.048(2) = 104.8, etc.).

Table 4. Parameters for potentials with barriers

N	g_s	r_s	r_C	c_1	b_1	c_1'	b_1'	D
1/2	2.6184	0.8545	0.3386	1.255	0.892	0.691	0.594	1.
1	2.7782	−0.4383	0.1487	−1.421(1)	4.545	1.048(2)	−2.327(1)	0.3373
2	2.7758	−0.2960	0.1580	−1.684(1)	5.337	1.605(2)	−3.894(1)	0.3288
3	2.7758	−0.2954	0.1581	−1.685(1)	5.340	1.605(2)	−3.899(1)	0.3285
∞	2.7758	−0.2954	0.1581	−1.685	5.340	1.605(2)	−3.899(1)	0.3285

Note: The case $N = \infty$ corresponds to the ECSCP (15), and finite N values correspond to the potentials (15). Here and in Table 5, the barrier penetrability D is calculated according to (16).

value [according to (3a), it is equal to unity] more slowly. On the other hand, the ratio r_C/r_s and the coefficients c_0, b_1, ... [which determine the Coulomb corrections in (5) and (7)] vary only slightly with increasing n.

5. POTENTIALS WITH A BARRIER

We have thus far assumed that the potential in (2) satisfies the condition $v(x) \geq 0$ everywhere in the interval $0 < x < \infty$. This corresponds to attractive potentials. However, potentials with a barrier are also often encountered in physics. For example, the potential

$$V(r) = -\frac{g}{r}e^{-\mu r}\cos\mu r, \qquad (15)$$

is considered in solid-state and plasma physics (see, for example, [25 - 29] and references therein). In the literature, this potential is referred to as the exponential cosine screened Coulomb potential (ECSCP). Numerical calculations for this potential yield the parameter values presented in Table 4, the effective range r_s being negative. According to [30], this is always the case, provided that the penetrability of the barrier in $V_s(r)$ is low. In the ECSCP, the barriers occur in the regions $\pi/2 < x \equiv \mu r < 3\pi/2$, $2.5\pi < x < 3.5\pi$, ...; in general, the nth barrier occurs in the region $a_n < x < b_n$, where $a_n = 2\pi(n-3/4)$ and $b_n = 2\pi(n-1/4)$.

The barrier penetrability for particles with zero energy is given by[6)]

$$D = \exp\left\{-g^{1/2}\int_0^\infty [v_-(x)]^{1/2}dx\right\}, \qquad (16)$$

where $v_-(x) = 0$ for $v(x) \geq 0$ and $v_-(x) = -v(x)$ for $v(x) < 0$. In the case under study, we have

$$c_1 n^{-1/2}e^{-n\pi} < I_n < c_2 n^{-1/2}e^{-n\pi}, \quad I_n \equiv \int_{a_n}^{b_n}\sqrt{-v(x)}dx,$$

where c_1 and c_2 are constants that are independent of n. For this reason, the penetrability D is finite, despite the fact that there are an infinite number of barriers in the ECSCP.

It should be noted that the coefficients b_1' and c_1' for the ECSCP are 1 to 2 orders of magnitude greater than those for the monotonic potentials from Table 1. This is indicative of the abrupt enhancement of Coulomb corrections to the effective range.

To verify this result, we performed calculations for the ECSCP truncated at the Nth barrier; that is, we set

$$v_N(x) = e^{-x}\frac{\cos x}{x}\theta(b_N - x), \quad b_N = 2\pi(N-1/4). \qquad (17)$$

The results of these calculations are presented in Table 4. It can be seen that even at $N = 1$ (only one barrier), the effective range becomes negative, while the coefficients b_1' and c_1' of the Coulomb corrections abruptly increase. Beginning at $N = 2$, all the quantities assume values that are rather close to those corresponding to $N = \infty$ [potential (15)]. On the other hand, we have $v(x) \geq 0$ at $N = 1/2$, and the quantities r_s, r_C, etc., do not exhibit qualitative features that distinguish them from the analogous quantities for other attractive potentials (cf. Table 1).

Table 5

a	g_s	r_s	c_1'	b_1'	D
10	1.8625	1.877	−0.093	0.849	0.996
8	1.9131	1.804	−0.023	0.830	0.986
6.5	1.9741	1.714	0.108	0.796	0.965
5	2.0780	1.552	0.511	0.692	0.908
4	2.1995	1.348	1.532	0.438	0.821
3	2.4238	0.926	8.774	−1.282	0.649
2	2.9548	−0.392	2.398(2)	−4.91(1)	0.340
1.5	3.5770	−2.847	1.993(1)	−2.55	0.151
1.2	4.2493	−7.622	1.249(1)	−0.885	5.98(−2)
1	4.9362	−1.737(1)	1.165(1)	−0.354	2.20(−2)
0.7	6.6880	−1.130(2)	1.60(1)	−4.74(−2)	1.38(−3)

[6)] Here, we do not take into account the reflection of particles between the neighboring barriers; of course, this can affect the preexponential factor.

Thus, the presence of a barrier may lead to a considerable increase in the Coulomb corrections to the effective range. This conclusion was verified not only for the ECSCP, but also for the model potential

$$v(x) = e^{-x}\left(\frac{1}{x} - \frac{1}{a}\right), \quad a > 0. \tag{18}$$

By varying the parameter a, we can easily change the barrier penetrability D (see Appendix B). This example confirms that Coulomb corrections increase abruptly when the barrier penetrability becomes small. In the case under study, this occurs for $D \lesssim 0.3$ (see Table 5). It would be interesting to study the physical implications of this result in problems involving the potential (15).

6. CONCLUSION

We calculated Coulomb corrections in low-energy scattering [that is, the coefficients c_0, c_1, etc., in equations (5) and (7)] for various model potentials $V_s(r)$.

The main conclusion that can be drawn from these calculations is that the most important corrections (c_0, c_1, and c_1') are weakly dependent on the form of the strong-interaction potentials and that they are determined primarily by the ratio $\rho = r_C/r_s$ that corresponds to the emergence of the s level in the system. This circumstance (which could not be anticipated) is useful in applications, because it enables us to choose a model potential that is most close to some realistic nucleon–nucleon potential (Hamada–Johnston, Reid, Paris potentials, etc.). These potentials are complicated and involve a large number of parameters, which naturally hampers numerical calculation of Coulomb corrections, especially qualitative analysis of the dependence of these corrections on the form of potential, on its small-distance behavior, etc. Moreover, nucleon–nucleon and nucleon–nucleus interaction potentials are not known at present. For this reason, we believe that in the calculation of Coulomb corrections, complicated realistic potentials do not possess substantial advantage over an appropriately chosen model potential.

We considered only real-valued potentials $V_s(r)$. At the same time, absorption is known to be always present in Coulomb systems involving additionally short-range forces (for example, in hadronic atoms). Nonetheless, the conditions

$$\beta_0 \equiv \mathrm{Im}(a_B/a_{cs}) \ll 1, \quad \mathrm{Im}\, r_s \ll |\mathrm{Re}\, r_s|, \tag{19}$$

hold in a number of cases in the resonant region; that is, absorption is weak (in particular, analysis performed in [12, 31] showed that in the dt system, $\beta_0 \approx 0.018$ and $r_{cs} \approx 5.0 - i0.3$ fm). Moreover, in contrast to $\mathrm{Re}(1/a_{cs})$, the imaginary part $\mathrm{Im}(1/a_{cs})$ is virtually constant in the vicinity of a resonance. Therefore, Coulomb corrections associated with the real part of potential are most important, because they can qualitatively change the situation. For example, it follows from (5) that at $a_s = \infty$ (exact resonance), the scattering length a_{cs} is completely determined by Coulomb corrections.

The following comment on potentials with barriers is in order. The fact that at low barrier penetrability, the effective range r_s is negative, with an absolute value satisfying the condition $|r_s| \gg r_N$, is well known [30]. It was shown above that in this case, Coulomb corrections to the effective range r_s also abruptly increase (by several orders of magnitude). It would be interesting to study the implications of this behavior for the problems of solid-state physics, where the potential (15) is often used.

Of problems that were not discussed in this study, two are worthy of special note. The first is the study of the effect exerted by the repulsive core in $V_s(r)$ on Coulomb corrections. The second is the extension of the results obtained here to the case of nonzero l – primarily to p states (here, Coulomb corrections to the effective range are of particular interest [11, 19]). We are going to address these issues in the future.

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APPENDIX A

Here, we present the formulas that describe the Coulomb corrections to a_s and r_s and which were used to calculate the quantities presented in Tables 1 and 2.

By $\tilde{\chi}_0$ and χ_0, we denote the wave functions corresponding to the emergence of the s level in the system with and without allowance for Coulomb interaction, respectively. The corresponding coupling constants are g_{cs} and g_s. We have

$$g_{cs} - g_s = \beta^{-1}\frac{r_N}{a_s} + \ldots, \quad \beta = \int_0^\infty v(x)\chi_0^2(x)dx, \tag{A.1}$$

where a_s is given by equation (1) at $a_{cs} = \infty$. The function $\tilde{\chi}_0(x)$ satisfies the equations

$$\frac{d^2\tilde{\chi}_0}{dx^2} + \left[g v(x) - \sigma\frac{r_N}{a_B x}\right]\tilde{\chi}_0(x) = 0, \quad x = r/r_N, \tag{A.2}$$

$$\tilde{\chi}_0(0) = 0, \quad \tilde{\chi}_0 \equiv \xi_0(x) = 1 + 2\sigma\frac{r_N}{a_B}x \times\left[\ln\frac{r_N}{a_B}x + 2C + \ln 2 - 1\right] + \ldots, \quad x \longrightarrow \infty. \tag{A.3}$$

The function $\xi_0(x)$ has the form

$$\xi_0(x) = \begin{cases} zK_1(z), & \sigma = +1 \\ 1, & \sigma = 0 \\ -\frac{\pi}{2} zN_1(z), & \sigma = -1, \end{cases} \qquad (A.4)$$

where $z = (8r_N x/a_B)^{1/2}$ and K_1 and N_1 are a Macdonald and a Neumann function. The equations for $\chi_0(x)$ are obtained from (A.2) and (A.3) in the limit $\sigma \longrightarrow 0$.

There are corrections of two types to the wave function χ_0. Corrections of the first type stem from Coulomb interaction (small parameter r_N/a_B), while corrections of the second type are due to the deviation of the s-level energy from zero (parameter r_N/a_s). In accordance with this, we can write

$$\tilde{\chi}_0(x) = \chi_0(x) + \frac{2r_N}{a_B}\varphi_1(x) + \beta^{-1}\frac{r_N}{a_s}\chi_1(x), \qquad (A.5)$$

where we discarded terms of orders $(r_N/a_B)^2$ and $r_N^2/a_B a_s$, as well as higher-order corrections. The functions φ_1 and χ_1 satisfy the nonhomogeneous equations

$$\begin{aligned} \varphi_1'' + g_s v(x)\varphi_1 &= \chi_1/x, \\ \chi_1'' + g_s v(x)\chi_1 &= v(x)\chi_0, \end{aligned} \qquad (A.6)$$

and are regular at the origin [$\varphi_1(0) = \chi_1(0) = 0$]. At infinity, these functions behave as

$$\begin{aligned} \varphi_1(x) &= \beta x + o(1), \\ \chi_1(x) &= x[\ln(x/x_C) - 1] + o(1), \end{aligned} \qquad (A.7)$$

where $x_C = r_C/r_N$ [see equation (6) and Fig. 3]. The absence of constant terms from (A.7) is a nontrivial circumstance; it is the condition that unambiguously determines the functions φ_1 and χ_1. Figure 3 shows these functions calculated numerically for the Yukawa and Hulthén potentials. It can be seen that for $x \geq 3$, they approach the asymptotic expressions in (A.7) and that for smaller x, they have one zero each [with the exception of the function $\chi_1(x)$ for the Hulthén potential; this function behaves as $\chi_1(x) \propto x^2$ for $x \ll 1$ and is positive for all x]. It can be shown [12] that for the ground state, the functions χ_1 and φ_1 can have no more than one zero in the interval $0 < x < \infty$.

Considering that the exact formula for the effective range corresponding to the emergence of the s level is [3, 31]

$$r_{cs} = 2r_N \int_0^\infty [\xi_0^2(x) - \tilde{\chi}_0^2(x)]\,dx, \qquad (A.8)$$

and using (A.3) and (A.5), we arrive at expansion (7). The Coulomb corrections to the scattering length and to the effective range are expressed in terms of the functions χ_0, χ_1, and φ_1. The explicit expressions for these

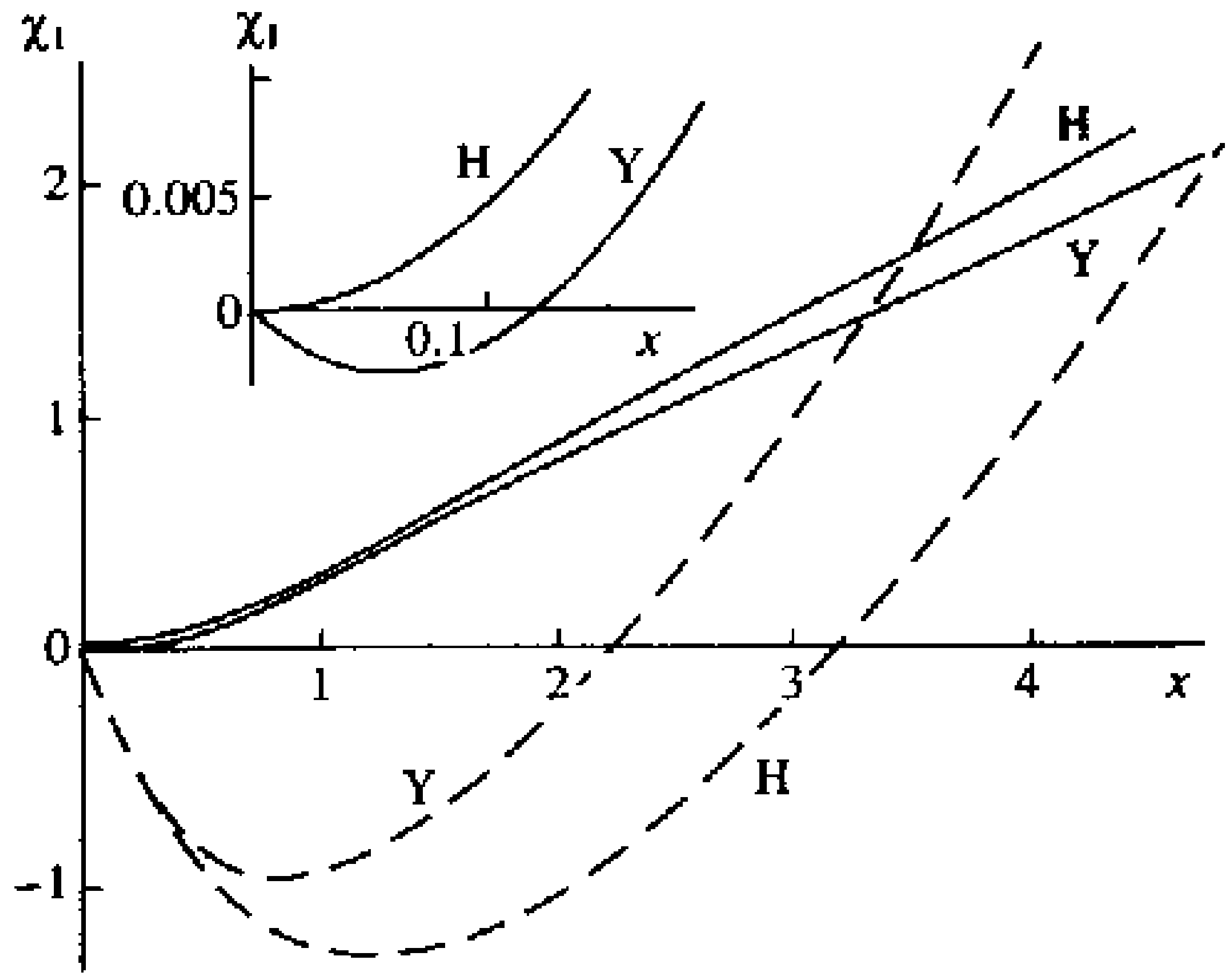

Fig. 3. Functions χ_1 (solid curves) and φ_1 (dashed curves) for the Yukawa (Y) and Hulthén (H) potentials (see nos. 1 and 4 in Table 1).

corrections are given in [10, 11]. Here, we represent these expressions in the more compact form

$$\{b_1, b_1'\} = \frac{2}{x_s}\int_0^\infty \{1, 2x\}\,\xi(x)\,dx, \qquad (A.9)$$

$$c_1 = \frac{2}{x_s}\int_0^\infty \eta(x)\,dx, \quad c_1' = \frac{8}{x_s^2}\int_0^\infty \eta(x)x\,dx, \qquad (A.10)$$

where

$$\xi(x) = 1 - \frac{\chi_0\chi_1}{\beta x}, \quad \eta(x) = 1 - \ln\frac{x}{x_C} + x^{-1}\chi_0\varphi_1. \qquad (A.11)$$

For short-range potentials, the functions defined in (A.11) exponentially decrease for $r \gg r_N$.

Let us make some comments on our numerical procedure. For the majority of potentials under study, we have

$$v(x) = \zeta x^{-1} + v_0 + v_1 x + \ldots, \quad x \longrightarrow 0 \qquad (A.12)$$

($\zeta = 1$ or 0). To circumvent difficulties associated with the singularity at $x = 0$, we began the integration of equations (3) and (A.6) from certain small x_0. For $0 < x \leq x_0$, we have

$$\chi_0(x) = a_1 x + a_2 x^2 + a_3 x^3 + O(x^4). \qquad (A.13)$$

From equation (3), it immediately follows that

$$a_2 = -\frac{1}{2}g_s\zeta a_1, \quad a_3 = -\frac{1}{6}g_s(\zeta a_2 + v_0 a_1), \ldots. \qquad (A.14)$$

Similar relations are obtained for χ_1 and φ_1. For sufficiently large x, we can set $v(x) \equiv 0$. Hence, as in the case of finite potentials, we assume that, for $x > x_\infty$, the required functions are given by

$$\chi_0(x) \equiv 1, \quad \chi_1(x) \equiv \beta x, \quad \varphi_1(x) \equiv x\,[\ln(x/x_C) - 1]\,.$$

In the interval $x_0 < x < x_\infty$ the equation was solved numerically by means of the standard routine. In calculating x_s, x_C, etc., the contribution from the region $x < x_0$ was taken into account explicitly. This can easily be done with the aid of expansion (A.13). For example, we have

$$\frac{1}{2}x_s = \int_{x_0}^{x_\infty}(1-\chi_0^2)\,dx + x_0 - \frac{1}{3}a_1^2x_0^3 - \frac{1}{2}a_1a_2x_0^4 - \frac{1}{5}(a_2^2+2a_1a_3)\,x_0^5. \qquad (A.15)$$

Let us also describe the procedure that we used to calculate the functions $\chi_1(x)$ and $\varphi_1(x)$. Specifying arbitrary initial values $\tilde{\chi}_1(x_0)$ and $\tilde{\chi}_1'(x_0)$, we sought a particular solution $\tilde{\chi}_1(x)$ of the nonhomogeneous equation (A.6). The required solution is $\chi_1 = \tilde{\chi}_1 - c_1\chi_0$, where

$$c_1 = \tilde{\chi}_1(x_\infty) - \beta x_\infty. \qquad (A.16)$$

In a similar way, we obtain

$$\varphi_1(x) = \tilde{\varphi}_1 - c_2\chi_0, \qquad c_2 = \tilde{\varphi}_1(x_\infty) - [\ln(x_\infty/x_C) - 1]\,, \qquad (A.17)$$

where $\tilde{\chi}_1\,(\tilde{\varphi}_1)$ is a particular solution. Here, we considered that the function $\chi_0(x)$ is a solution of the corresponding homogeneous equation.

In calculating the integrals for b_1, c_1, and other coefficients [see (A.9) - (A.11)], we took into account the contribution from the region $x < x_0$ by means of the same method as that used to obtain (A.15). The points x_0 and x_∞ were varied, and values that were stabilized as the result of this procedure were considered reliable. The accuracy achieved for all the quantities included in Tables 1 and 2 was not poorer than 10^{-5}.

For the sake of comparison with the results of previous calculations [11], we note that

$$b_1' = k_0/\beta x_s, \quad c_1' \equiv h_0, \qquad (A.18)$$

where k_0 and h_0 are coefficients introduced in [11].

APPENDIX B

The potential (18) has a Coulomb singularity at the origin, corresponds to attraction in the region $0 < x < a$, and possesses a barrier for $x > a$ [$x_m = 1/2(a + \sqrt{a^2+a})$]. Up to a preexponential factor, the barrier penetrability for particles with zero energy is given by

$$D = \exp\{-g^{1/2}I(a)\}, \qquad (B.1)$$

where

$$I(a) = 2\int_a^\infty \sqrt{-v(x)}\,dx = 2(\pi/a)^{1/2}e^{-a/4}W_{-1/2,\,1/2}(a/2), \qquad (B.2)$$

and W is a Whittaker function.[7] In the limiting cases, we obtain

$$x_m = 1/2(\sqrt{a} + a + \ldots), \qquad I = 2a^{-1} + 1/2\ln a + O(1), \quad a \longrightarrow 0, \qquad (B.3)$$

$$x_m = a + \frac{1}{4} - \frac{1}{8a} + \ldots, \qquad I = \sqrt{8\pi}e^{-a/2}\left(\frac{1}{a} - \frac{3}{2a^2} + \ldots\right), \qquad (B.4)$$

$$v_m = v(x_m) \propto a^{-2}\exp(-a), \quad a \longrightarrow \infty.$$

Thus, the barrier has an exponentially small penetrability for small a [$D \propto \exp(-2\sqrt{g/a})$] and virtually disappears for $a \gg 1$ ($D \longrightarrow 1$). With decreasing barrier penetrability, the effective range also decreases: it vanishes at $a = a_0 \simeq 2.2$ and is negative for $a < a_0$, in agreement with the results presented in [30]. The coefficients c_1' and b_1' are anomalously large at $a = a_0$ (here, b_1' changes sign). Analysis of this example enabled us to qualitatively clarify the dependence of Coulomb corrections on the barrier penetrability.

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